Induction equation and Gauss coefficients

Ankit Barik

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1 Divergence and Curl

These operators act on *vector fields*. What is a vector field? It is simply a space where each point is assigned a vector. For example, if I specify the components of a vector field $\boldsymbol{A}$ as,

$$\boldsymbol{A} = A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}} + A_z\hat{\boldsymbol{k}}, \tag{1}$$

and write down the components as functions of coordinates $A_x(x, y, z)$, $A_y(x, y, z)$, $A_z(x, y, z)$, then one can define what the vector $\boldsymbol{A}$ looks like at each point in space. $\boldsymbol{A}$ is then called a *vector field*. We encounter vector fields all the time in physics - could be fluid velocity, electric field, magnetic field etc.

The upward down triangle operator (∇), otherwise known as the 'del' or 'nabla' operator can be written down in cartesian coordinates as

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \tag{2}$$

This makes it easy to remember the operations of divergence and curl where this operator kind of behaves like a vector.

Divergence

In terms of this operator, the **divergence** of a vector field can be written down as,

$$\nabla \cdot \boldsymbol{A} = (\frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}) \cdot (A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}} + A_z\hat{\boldsymbol{k}}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{3}$$

Note that, the divergence of a vector field gives a *scalar field* (keep in mind that 'field' simply means that stuff varies in space and is a function of coordinates). The divergence of a vector field tells you by how much the components change when moved in the corresponding direction. Now, imagine you are near a 'source'. The vectors will diverge out from that point, leading to a positive gradient in all directions moving away from that point. This consequently leads to a positive divergence. Thus, a positive *divergence*, as the name signifies, tells you that stuff is diverging away from that point and there is a source. The opposite is true for a 'sink'. This is further illustrated in figure 1.

In spherical coordinates (r, θ, ϕ), this operator looks like below:

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{4}$$

Curl

In terms of the ∇ operator, the **curl** of a vector field can be written as,

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{k}} \tag{5}$$

The curl of a vector field gives another vector field that denotes the direction and magnitude of rotation of the first vector field. This is shown in figure 2.

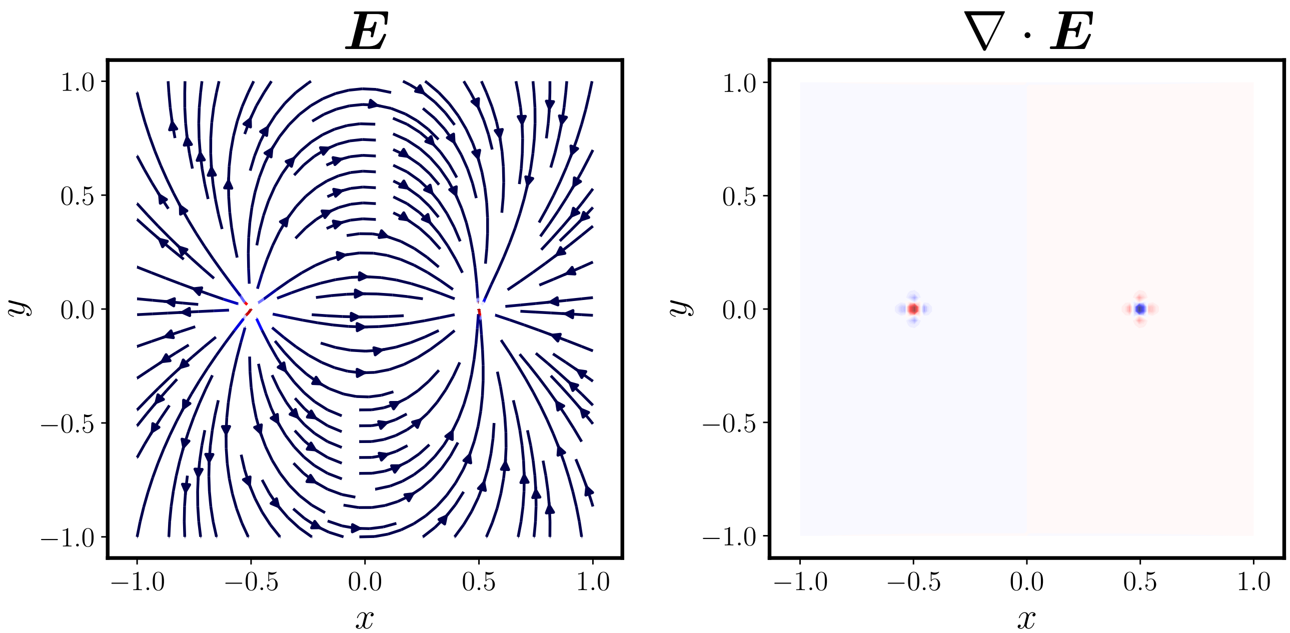


Figure 1: Electric field due to two charges, positive (source) and negative (sink). The divergence clearly shows the charges.

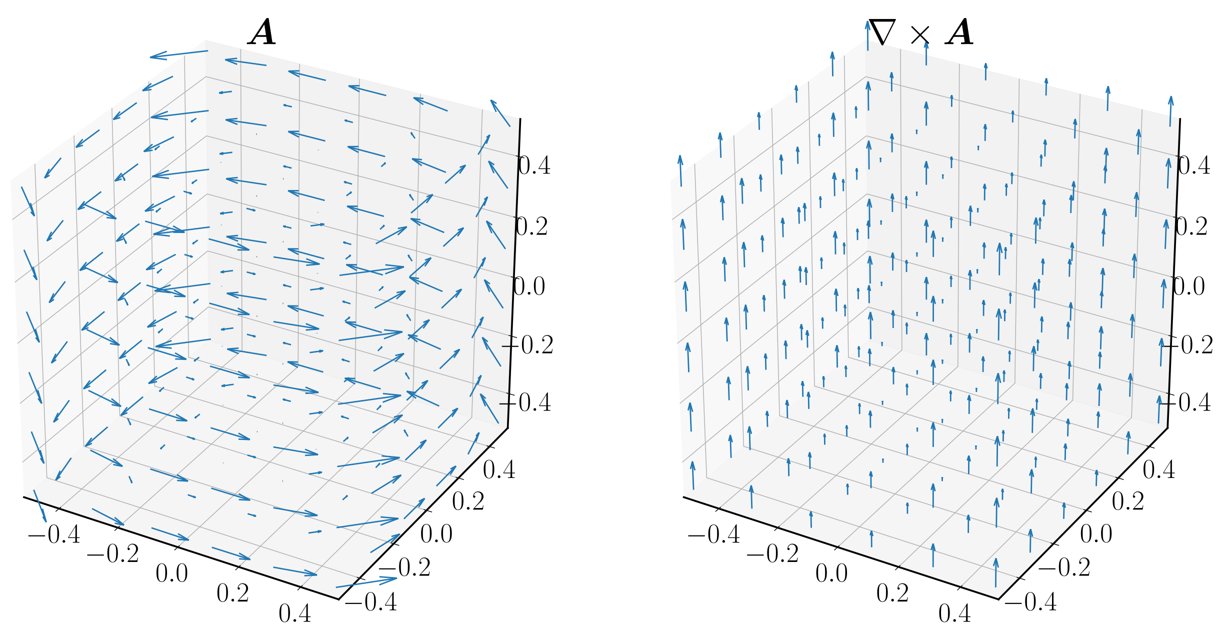


Figure 2: Curl of a vector field $\boldsymbol{A}$. The curl on the right hand side shows the direction of rotation, the arrow lengths showing the magnitude. Note how the arrow lengths increase outward for both plots.

2 Induction equation

To understand how dynamos operate, it might be useful to have an equation that tells us how the magnetic field evolves due to various effects that act on it. We will first begin by writing down Maxwell's equations of electromagnetism,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{6}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{7}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0}, \tag{8}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{9}$$

It can be shown that while dealing with velocities which are really slow compared to the speed of light, we can ignore displacement currents ($1/c^2(\partial \boldsymbol{E}/\partial t)$) and thus, we end up with what are known as 'pre-Maxwell' equations,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{10}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{11}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0}, \tag{12}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{13}$$

Lastly, we need one more equation to obtain the desired induction equation. This is given by Ohm's law,

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}), \tag{14}$$

where $\boldsymbol{u}$ is the velocity of the moving frame (the fluid velocity in our case of studying dynamos).

Why do we ignore displacement current?

If the typical strengths of electric and magnetic fields are E^* and B^* then using $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$, we can write

$$\frac{E^*}{L} \sim \frac{B^*}{\tau},$$

$$\Rightarrow E^* \sim \frac{B^* L}{\tau},$$

where L and τ are typical timescales of motion. Consider the terms,

$$|\nabla \times \boldsymbol{B}| \sim \frac{B^*}{L},$$

and

$$\frac{1}{c^2}\left|\frac{\partial \boldsymbol{E}}{\partial t}\right| \sim \frac{1}{c^2}\frac{E^*}{\tau} = \frac{1}{c^2}\frac{B^* L}{\tau^2}.$$

The ratio is

$$\frac{1/c^2|\partial \boldsymbol{E}/\partial t|}{|\nabla \times \boldsymbol{B}|} \sim \frac{L^2/\tau^2}{c^2} = \frac{u^2}{c^2} \ll 1,$$

where, u is a typical velocity scale of astrophysical fluids and is much less than the speed of light.

Where does the $\boldsymbol{u} \times B$ come from in Ohm's law?

If you are a charged particle in a frame moving with velocity $\boldsymbol{u}$ in the presence of a magnetic field $\boldsymbol{B}$, the force acting on you is,

$$F_L = q\boldsymbol{u} \times \boldsymbol{B},$$

where q is the charge on the particle. Thus, it is equivelent to saying that there is an electric field $\boldsymbol{E_L} = \boldsymbol{u} \times \boldsymbol{B}$ acting on the charged particle. Thus, if an electric field $\boldsymbol{E}$ is present at the same time, the resultant electric field becomes

$$\boldsymbol{E}_r = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}.$$

To derive an equation for the evolution of magnetic field in a dynamo, we need an equation for $\dfrac{\partial \boldsymbol{B}}{\partial t}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{B}$. There are two ways to obtain it.

Method I

From (11), we obtain,

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial t} &= -\nabla \times \boldsymbol{E} \\ &= -\nabla \times \left(\frac{\boldsymbol{J}}{\sigma} - \boldsymbol{u} \times \boldsymbol{B} \right) \\ &= -\nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B} - \boldsymbol{u} \times \boldsymbol{B} \right), \end{aligned} \tag{15}$$

where we have used (14) in the second step and (12) in the third step. Thus,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{u} \times \boldsymbol{B} - \frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B} \right). \tag{16}$$

Method II

From (12), we obtain,

$$\begin{aligned} &\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} = \mu_0 \sigma (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \\ \Rightarrow &\frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}. \end{aligned} \tag{17}$$

where we have used Ohm's law (14). To obtain an equation for $\dfrac{\partial \boldsymbol{B}}{\partial t}$, we need to use (11). To do so, we take the curl ($\nabla \times$) of both sides, and get

$$\begin{aligned} &\nabla \times \frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B} = \nabla \times \boldsymbol{E} + \nabla \times \boldsymbol{u} \times \boldsymbol{B} \\ \Rightarrow &\nabla \times \frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B} = -\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{u} \times \boldsymbol{B}. \end{aligned} \tag{18}$$

Rearranging the above, we get,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{u} \times \boldsymbol{B} - \frac{1}{\mu_0 \sigma} \nabla \times \boldsymbol{B} \right). \tag{19}$$

Object	$\lambda (m^2/s)$	$L(m)$	$U(m/s)$	Rm
Earth's core	2	3×10^6	3×10^{-4}	450
Star	0.5	10^9	1	2×10^9
Cu sphere	0.15	1	1	6

Table 1: Typical Rm values for various objects. Think what it might be for a galaxy.

The quantity $1/\mu_0\sigma$ is referred to as 'magnetic diffusivity' and can is denoted by either λ or η depending on who writes it. In these notes, I will use λ. Thus, the equation for the evolution of magnetic field is given by the **induction equation**,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \lambda \nabla \times \boldsymbol{B}) \,. \tag{20}$$

Special case: when λ is a constant

In the case when λ can be assumed to be a constant, the induction equation can be written as,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \lambda \nabla \times \nabla \times \boldsymbol{B}. \tag{21}$$

Using the vector identity, $\nabla \times \nabla \times = \nabla(\nabla \cdot) - \nabla^2$ and $\nabla \cdot \boldsymbol{B} = 0$, we obtain,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \lambda \nabla^2 \boldsymbol{B}. \tag{22}$$

We will take λ to be constant for now. In case not, there is an extra term of $(-\nabla\lambda \times \nabla \times \boldsymbol{B})$ on the RHS.

The first term in equation (22) determines the interaction between fluid flow and magnetic field and contributes to production of magnetic field. The second term is responsible for Ohmic dissipation. Note that the dissipation term is similar to the one for heat. Taking the dimensional ratio of the two terms on the RHS, we get,

$$\frac{|\nabla \times (\boldsymbol{u} \times \boldsymbol{B})\,|}{|\lambda \nabla^2 \boldsymbol{B}|} = \frac{UB/L}{\lambda B/L^2} = \frac{UL}{\lambda}, \tag{23}$$

where, U, B and L are typical scales of fluid velocity, magnetic field and length, respectively. This quantity is defined as the **magnetic Reynolds number**, Rm,

$$Rm = \frac{UL}{\lambda} = UL\mu_0\sigma, \tag{24}$$

where we have used the definition of $\lambda = 1/\mu_0\sigma$. Thus, Rm depends on the typical flow length-scale, flow speed and conductivity of the fluid. Typical Rm values are given in table 1.

Using vector identities, the first term on the right hand side in (22) can be expanded and the equation can be rewritten as

$$\underbrace{\frac{\partial \boldsymbol{B}}{\partial t}}_{\text{Change in time}} = \underbrace{\boldsymbol{B} \cdot \nabla \boldsymbol{u}}_{\text{Stretching}} \underbrace{- \boldsymbol{u} \cdot \nabla \boldsymbol{B}}_{\text{Advection}} \underbrace{- \boldsymbol{B}(\nabla \cdot \boldsymbol{u})}_{\text{Expansion/compression}} + \underbrace{\lambda \nabla^2 \boldsymbol{B}}_{\text{Dissipation}}, \tag{25}$$

where the various contributions have been labelled. Terms such as $\boldsymbol{u} \cdot \nabla \boldsymbol{B}$ refer to the *material derivative*, which essentially tell us how one a gradient of one quantity varies along the other. The first term tells us how magnetic fields are stretched and twisted by fluid motions. The second term tells us how the magnetic field is carried around ('advected') by the fluid motions, while the third term expresses how changes are induced in the magnetic field due to expansion or compression of the fluid at a point.

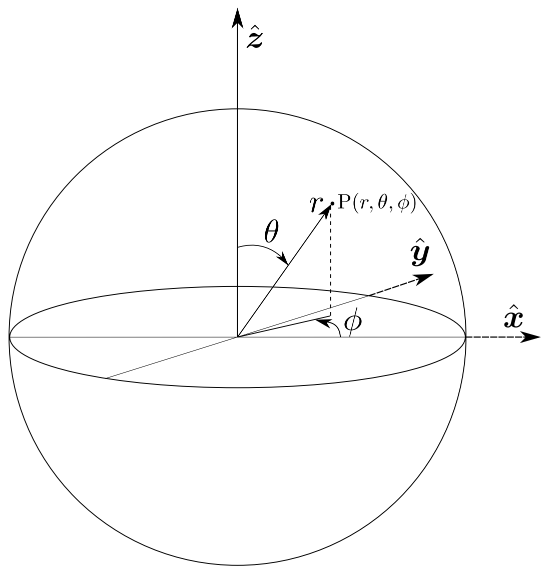


Figure 3: Spherical coordinates. r is the radial distance from origin, ϕ is the longitude, measure counterclockwise from x-axis, while θ is the colatitude, measure from the z-axis.

3 Solution to Laplace's equation

In an insulating medium, there are no electric currents, hence,

$$\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B} = \boldsymbol{0}. \tag{26}$$

which means $\boldsymbol{B}$ can be written in terms of the gradient of a scalar potential (since curl of a gradient is always zero),

$$\boldsymbol{B} = -\nabla V. \tag{27}$$

Thus, $\nabla \cdot \boldsymbol{B} = 0$ gives us

$$\nabla^2 V = 0. \tag{28}$$

Expanding this in spherical coordinates (r, θ, ϕ) (figure 3), we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta} V\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} V = 0. \tag{29}$$

To solve this, we will use separation of variables. We write the scalar $V(r, \theta, \phi)$ as a product of three functions

$$V = R(r)\Theta(\theta)\Phi(\phi). \tag{30}$$

Substituting and multiplying by r^2, we get

$$\Theta\Phi \frac{d}{dr}\left(r^2 \frac{d}{dr} R\right) + \frac{R\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\Theta\right) + \frac{R\Theta}{\sin^2\theta}\frac{d^2}{d\phi^2}\Phi = 0. \tag{31}$$

Dividing throughout by $V = R\Theta\Phi$ (assumption : $V \neq 0$ anywhere in the domain of study), we obtain,

$$\begin{aligned} &\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{d}{dr} R\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\Theta\right) + \frac{1}{\Phi \sin^2\theta}\frac{d^2}{d\phi^2}\Phi = 0 \\ \Rightarrow &\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2 \frac{d}{dr} R\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\Theta\right) = -\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi. \end{aligned} \tag{32}$$

We have two sides dependent on completely different independent variables and the two are equal. The only way this is possible is if both are equal to a constant. Let us call this constant m^2. Thus,

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2 \frac{d}{dr} R\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\Theta\right) = -\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi = m^2. \tag{33}$$

Solution for Φ

It is relatively easier to solve for Φ, using

$$\begin{aligned} -\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi &= m^2 \\ \Rightarrow \frac{d^2}{d\phi^2}\Phi + m^2\Phi &= 0. \end{aligned} \tag{34}$$

This is the familiar equation of a harmonic oscillator ($d^2x/dt^2 + \omega^2 x = 0$). Substituting $\Phi = Ae^{k\phi}$, we obtain

$$\begin{aligned} &k^2 + m^2 = 0 \\ \Rightarrow &k = \pm \mathrm{i}m. \end{aligned} \tag{35}$$

Thus, solutions for Φ are of the form $Ae^{\pm \mathrm{i}m\phi}$. The general solution would thus, be a linear combination of the possible solutions,

$$\Phi(\phi) = C_1 e^{\mathrm{i}m\phi} + C_2 e^{-\mathrm{i}m\phi}. \tag{36}$$

Note : This is just for a single m. We will eventually have to sum over different m in order to obtain the final general solution.

How does the solution look like? We can expand the exponentials using Euler's identity ($e^{\mathrm{i}m\phi} = \cos(m\phi) + \mathrm{i}\sin(m\phi)$) to obtain an experession that looks as follows,

$$\Phi(\phi) = C_1' \cos(m\phi) + C_2' \sin(m\phi), \tag{37}$$

which is a sinusoidal solution, but in the longitudinal direction. It is equivalent to wrapping a cosine or a sine function around a circle. The periodicity of a simple cosine or sine is 2π. Hence, that of a $\cos(m\phi)$ or $\sin(m\phi)$ is $2\pi/m$, which means that if you rotate the function by $1/m^{\text{th}}$ part of a circle, you'll get the same thing. This is shown in figure 4 which shows $\cos(m\phi)$ being wrapped around a circle ($0 \leqslant \phi \leqslant 2\pi$) for different m values. Note that addition of a sine component will only change the phase and amplitude (rotate the plots and make the maximum bigger or smaller), and not the shape of the functions.

Solution for Θ

Going back to (33),

$$\begin{aligned} \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right) &= m^2 \\ \Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) = \frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right) &- \frac{m^2}{\sin^2\theta}, \end{aligned} \tag{38}$$

we again have two sides which are dependent on completely independent variables. For this to be true for any arbitrary value of the variables, both sides need to equal a constant. Let us write this constant as $-l(l+1)$ (for mathematical convenience later),

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) = -\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right) + \frac{m^2}{\sin^2\theta} = l(l+1). \tag{39}$$

Now, let us solve for Θ as follows,

$$\begin{aligned} -\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right) + \frac{m^2}{\sin^2\theta} &= l(l+1) \\ \Rightarrow \frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)\Theta &= 0. \end{aligned} \tag{40}$$

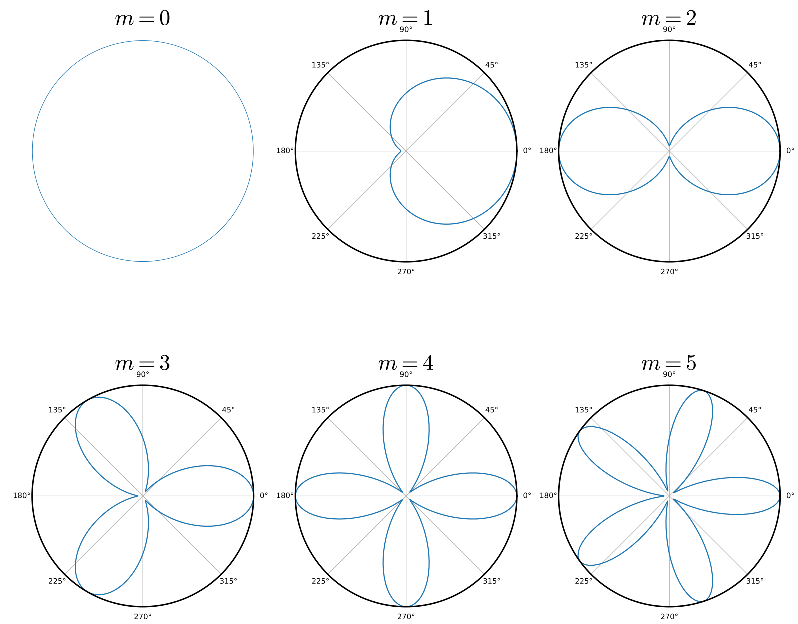


Figure 4: Example of how $\cos(m\phi)$ looks like wrapped around a circle. Note that $m = 0$ only represents a constant $= 1$.

l	m	$P_l^m(x)$	$P_l^m(\cos\theta)$
0	0	1	1
1	0	x	$\cos\theta$
1	1	$\sqrt{1-x^2}$	$\sin\theta$
2	0	$3x^2-1$	$3\cos^2\theta-1$
2	1	$x\sqrt{1-x^2}$	$\cos\theta\sin\theta$
2	2	$1-x^2$	$\sin^2\theta$

Table 2: List of first few associated Legendre polynomials (not normalised).

This looks "nice" but is still a bit complicated to solve. In order to solve this, we need to work with a new variable, $x = \cos\theta$. Using chain rules, the derivatives now become

$$\begin{aligned} \frac{d}{d\theta} &= \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx} = -\sqrt{1-x^2}\frac{d}{dx}, \\ \frac{d^2}{d\theta^2} &= \frac{d}{d\theta}\left(-\sqrt{1-x^2}\frac{d}{dx}\right) \\ &= \frac{dx}{d\theta}\frac{d}{dx}\left(-\sqrt{1-x^2}\frac{d}{dx}\right) \\ &= -\sqrt{1-x^2}\left(-\sqrt{1-x^2}\frac{d^2}{dx^2} + \frac{x}{\sqrt{1-x^2}}\frac{d}{dx}\right) \\ &= (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}. \end{aligned} \tag{41}$$

Using the above expressions, (40) becomes,

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left(l(l+1) - \frac{m^2}{1-x^2}\right)\Theta = 0. \tag{42}$$

This is a well-known differential equation called the "general Legendre equation", whose solutions are the *associated Legendre polynomials*,

$$\Theta(\theta) = P_l^m(x) = P_l^m(\cos\theta). \tag{43}$$

The first few polynomials are listed in table 2. Note that l can take on any value greater than zero while $0 \le m \le l$. For more details you can visit the Wikipedia page. Note that the general definition has both positive and negative m. However, it can be shown that for real quantities, one need only consider $m \ge 0$.

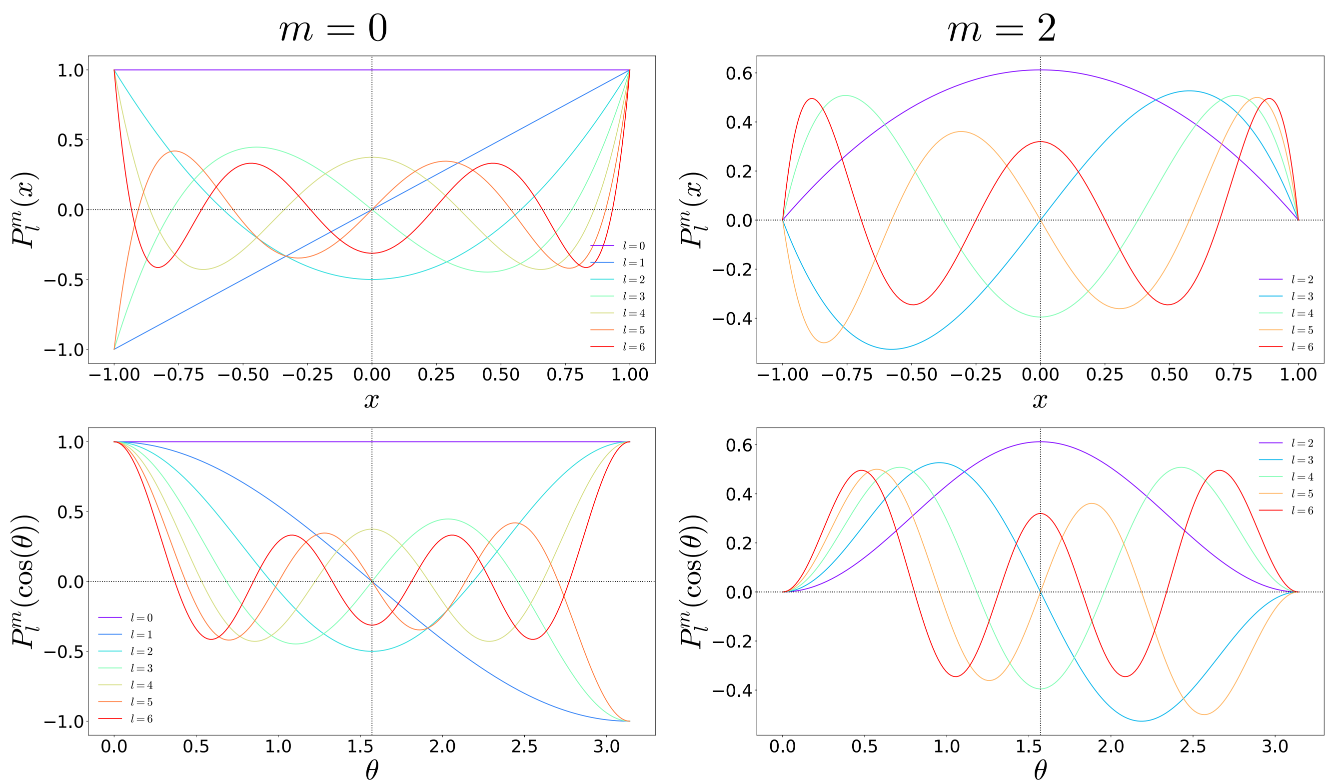


Figure 5: Plots of $P_l^m(x)$ and $P_l^m(\cos\theta)$ with respect to x and θ.

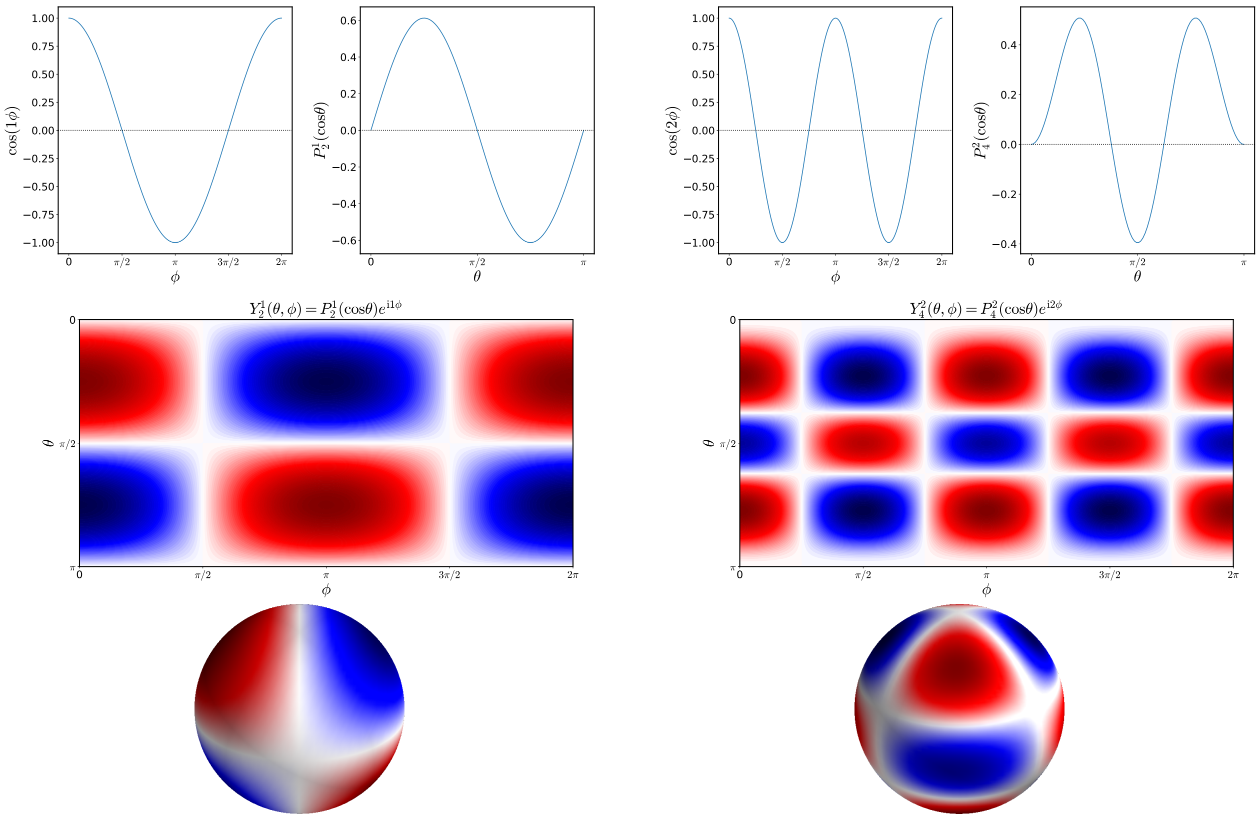
As one can see, in general, curves get more "squiggly" for a fixed m and higher l, and less "squiggly" for fixed l and higher m. If one counts the number of times a single curve crosses the zero line, one can find a relation between l, m and this number for a single curve (it's equal to $l-m$). In addition, remembering that the vertical lines at $x = 0$ or $\theta = \pi/2$ represent the equator, one can also say something about the equatorial symmetry of the curve (is it symmetric or anti-symmetric with respect to the equator?). It also depends on $l-m$. For a P_l^m which is symmetric (antisymmetric) w.r.t equator, $l-m$ =even (odd).

Spherical harmonics

We take a pause here to introduce a set of basis functions in 2D spherical coordinates (θ, ϕ). When we combine the solutions for $\Theta(\theta)$ and $\Phi(\phi)$, we obtain functions that look like

$$Y_l^m(\theta, \phi) = P_l^m(\cos\theta)e^{im\phi}. \tag{44}$$

These functions are called spherical harmonics with *degree* l and *order* m. How do they look like in physical space? Remember that the solution for Φ is a sine or cosine function wrapped around a circle, while that for Θ, the solutions are polynomials $P_l^m(\cos\theta)$ which have a sort of sinusoidal shape. Multiplying the two gives interesting patterns in 2D. This is shown in figure 6. The number of zeros in longitude is m, while the number of zeros (or nodes) in



(a) $Y_2^1 = P_2^1(\cos\theta)e^{i\phi}$ (b) $Y_4^2 = P_4^2(\cos\theta)e^{i2\phi}$

Figure 6: Structure of spherical harmonics. Each subfigure shows the solution for $\Phi(\phi) = \cos(m\phi)$, $\Theta(\theta) = P_l^m(\cos\theta)$ and finally the product, $Y_l^m(\theta, \phi)$.

longitude is given by $l - m$. Thus, larger the l, the larger is the possibility of having complex structures (larger $l - m$). A function $f(\theta, \phi)$ on a spherical surface can be written as,

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} c_{lm} Y_l^m(\theta, \phi). \tag{45}$$

Since spherical harmonics are orthogonal,

$$\int_0^{2\pi} \int_0^{\pi} Y_l^m(\theta, \phi) Y_{l'}^{m'*}(\theta, \phi) \sin\theta d\theta d\phi = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \delta_{ll'} \delta_{mm'}, \tag{46}$$

one can readily obtain the coefficients c_{lm} as,

$$c_{lm} = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) Y_l^{m*}(\theta, \phi) \sin\theta d\theta d\phi. \tag{47}$$

An example of such an expansion is shown in figure 7.

Solution for R

Having dealt with θ and ϕ directions, we now turn to the equation for R. Going back to (39), we get,

$$\begin{aligned} &\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{d}{dr} R \right) = l(l+1) \\ \Rightarrow &r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - l(l+1)R = 0. \end{aligned} \tag{48}$$

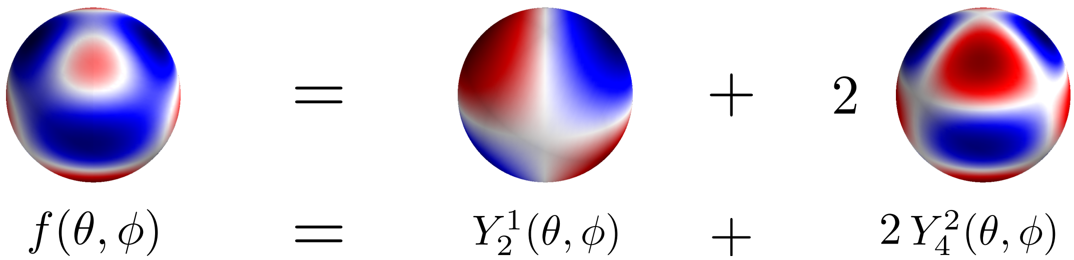


Figure 7: Example of a spherical harmonic expansion of a function $f(\theta, \phi)$ on a spherical surface.

This is a special kind of differential equation called *Euler-Cauchy equation*. The way to solve this is by guessing a solution that goes as $R = Ar^{\alpha}$. Substituting this, we get

$$\begin{aligned} &\alpha(\alpha-1)r^{\alpha} + 2\alpha r^{\alpha} - l(l+1)r^{\alpha} = 0 \\ \Rightarrow &\alpha^2 + \alpha - l(l+1) = 0 \\ \Rightarrow &(\alpha - l)(\alpha + l + 1) = 0 \\ \Rightarrow &\alpha = l, -l-1. \end{aligned} \tag{49}$$

Thus, the general solution is,

$$R(r) = D_{1l} r^{l} + D_{2l} r^{-l-1}. \tag{50}$$

We have two cases here:

Case I : Internal field

In the case of an internal source of field, the potential needs to be finite while $r \to \infty$, we need $D_{1l} = 0 \ \forall \ l$ and thus,

$$R(r) = D_{2l} r^{-l-1}. \tag{51}$$

Combining the different solutions from (36),(43), and (51), and summing over all possible l and m, one obtains the general solution for the potential V,

$$V = \sum_{l=1}^{\infty} \sum_{m=0}^{l} D_{2l} r^{-l-1} [C_1 e^{\mathrm{i}m\phi} + C_2 e^{-\mathrm{i}m\phi}] P_l^m(\cos\theta). \tag{52}$$

Expanding the exponentials using Euler's identity ($e^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x)$), one can write,

$$V = \sum_{l=1}^{\infty} \sum_{m=0}^{l} D_{2l} r^{-l-1} [g_{lm} \cos(m\phi) + h_{lm} \sin(m\phi)] P_l^m(\cos\theta). \tag{53}$$

g_{lm} and h_{lm} are called **Gauss coefficients**. We obtain this typically from observations of the radial magnetic field at the surface ($r = a$). Thus, the surface expansion will not have any factors of radius r. This gives a way to define the constant $D_{2l} = a^{l+1}$. Lastly, we include an extra factor of a so that the measured radial field component $B_r = -\partial V/\partial r$ does not have any extra factors and can be readily expanded from observations. Finally, the expression for V becomes

$$V_I = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l+1} [g_{lm} \cos(m\phi) + h_{lm} \sin(m\phi)] P_l^m(\cos\theta). \tag{54}$$

Case II : External field

In the case of an external source of field, the potential needs to be finite while $r \to 0$, we need $D_{2l} = 0 \ \forall \ l$ and thus,

$$R(r) = D_{1l} r^{l}. \tag{55}$$

Following the same twisted logic in case of internal field, we obtain the final expression to be,

$$V_E = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{r}{a}\right)^l [g_{lm} \cos(m\phi) + h_{lm} \sin(m\phi)] P_l^m(\cos\theta). \tag{56}$$

Note that $V_I(a) = V_E(a)$, as one would expect.

What do internal and external mean?

Remember that the terms *internal* and *external* only refer to where the source of the field is - is it below the radius of interest, or above? For example, if you are continuing the field at the surface of a planet down into the mantle, you would use equation (54) because the source of the field is below, in the core. Thus, for downward or upward continuation of magnetic field, (54) is the expression that is widely used and is commonly encountered in planetary science.