Inertial and gravity waves

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Waves vs Modes

- Waves are travelling, don't care about boundaries till they reflect. They are 'local' in nature.
- Modes are aware of boundaries and are 'global' in nature.

Types of waves in planetary fluid dynamics

Type of wave	Restoring force(s)
Acoustic (p-modes)	$-\nabla p$
Inertial	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u}$
Surface/Internal gravity (f-/g-modes)	$\rho' \boldsymbol{g}$
Inertia-gravity or gravito-inertial	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} + \rho' \boldsymbol{g}$
Alfvén	$\boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magnetoacoustic	$-\nabla p + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magneto-Coriolis (MC)	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magnetic, Archimedean, Coriolis (MAC)	$-2\boldsymbol{\Omega} \times \boldsymbol{u} + \rho' \boldsymbol{g} + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$

Inertial Modes

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} \qquad (1)$$

Inertial Modes

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BC: $\quad \boldsymbol{u} \cdot \hat{\boldsymbol{n}} = \boldsymbol{0}$

Inertial Modes

$$-\mathrm{i}\omega \boldsymbol{u} = -\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} \tag{2}$$

$$\text{BC:} \qquad \boldsymbol{u} \cdot \hat{\boldsymbol{n}} = \mathbf{0}$$

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$\omega = 0 \rightarrow$ Geostrophic mode (NOT an inertial mode)

Inertial Modes

Properties:

Denoting the solutions using $\boldsymbol{Q}_i$, one can prove

- $|\omega| \leq 2\Omega$
- Orthogonal: $\int \boldsymbol{Q}_m^\dagger \cdot \boldsymbol{Q}_n dV = \delta_{mn}$

Inertial Modes

- Solution depends on container
- Can be solved analytically in a full sphere (Zhang et al., 2001), and
- have the form $\boldsymbol{Q}(r,\theta)e^{\mathrm{i}(m\phi-\omega t)}$
- Discrete frequencies (unlike plane inertial waves)

Modes in a sphere

In a sphere, the frequencies satisfy:

$$\left(1-\frac{\omega^2}{4}\right)\frac{d}{d\omega}P_{lm}(\omega/2)=\frac{m}{2}P_{lm}(\omega/2) \tag{3}$$

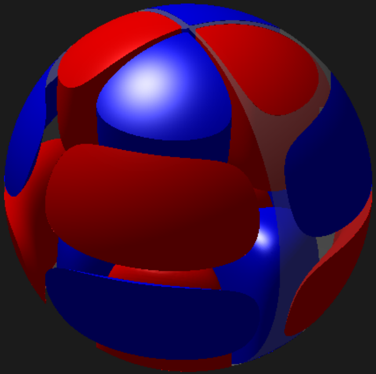
Simplified:

$$(lx+m)P_{lm}(x)=(l+m)P_{l-1,m}(x) \tag{4}$$

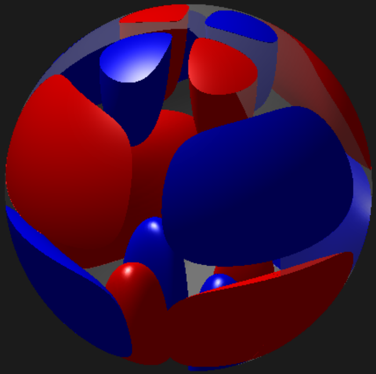
where, $x=\omega/2$

Number of solutions $= l-m-\nu_{lm}$, where $\nu_{lm}=\begin{cases}0, & \text{if } l-m \text{ is even}\\ 1, & \text{if } l-m \text{ is odd}\end{cases}$

Modes in a sphere



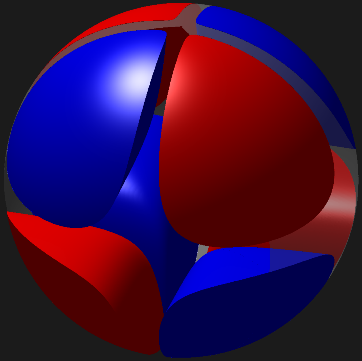
u_s



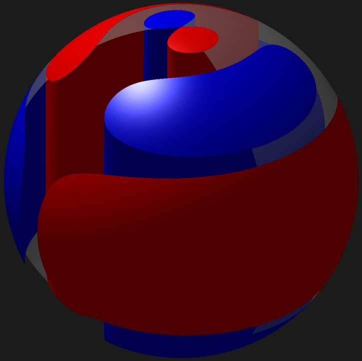
u_ϕ

$(5, 2)$ mode

Other examples



$(3, 2)$ mode



$(5, 1)$ mode

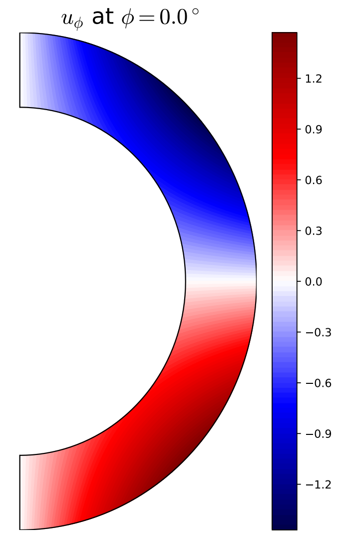
Internal shear layers

What happens when there are realistic boundary conditions?

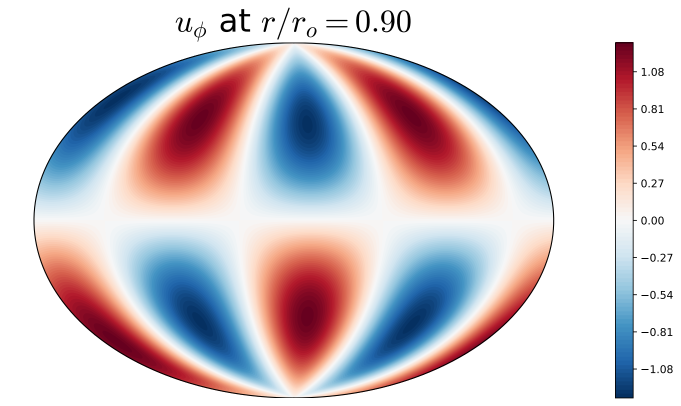
Internal shear layers

- Hyperbolic equation
- Normally require Cauchy BC, but we provide Dirichlet or Neumann
- We don't know, in advance, whether a solution even exists
- Discrete ω

Free slip : $(3, 2)$

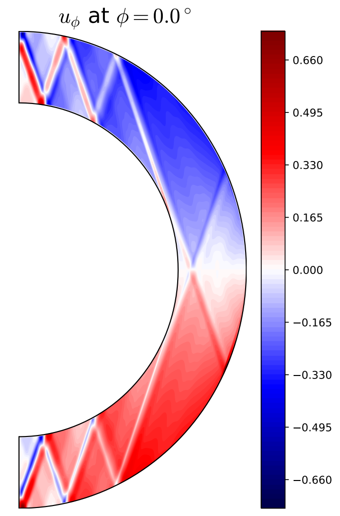


u_ϕ at $\phi = 0.0^\circ$
1.2
0.9
0.6
0.3
0.0
−0.3
−0.6
−0.9
−1.2

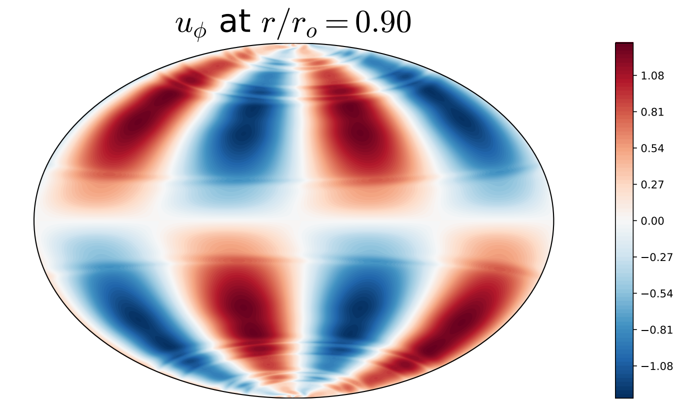


u_ϕ at $r/r_o = 0.90$
1.08
0.81
0.54
0.27
0.00
−0.27
−0.54
−0.81
−1.08

No slip : $(3,2)$

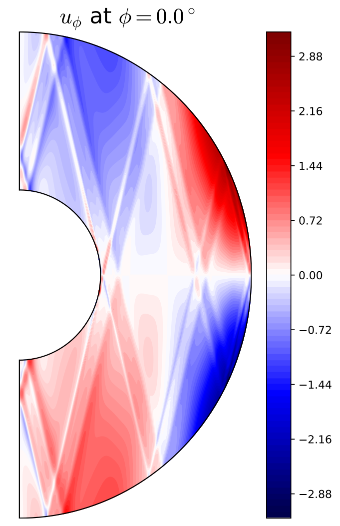


u_ϕ at $\phi = 0.0^\circ$
0.660
0.495
0.330
0.165
0.000
−0.165
−0.330
−0.495
−0.660

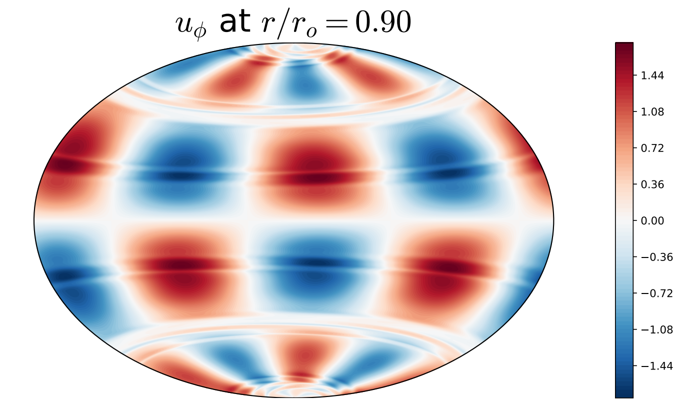


u_ϕ at $r/r_o = 0.90$
1.08
0.81
0.54
0.27
0.00
−0.27
−0.54
−0.81
−1.08

Free slip : $(5, 2)$

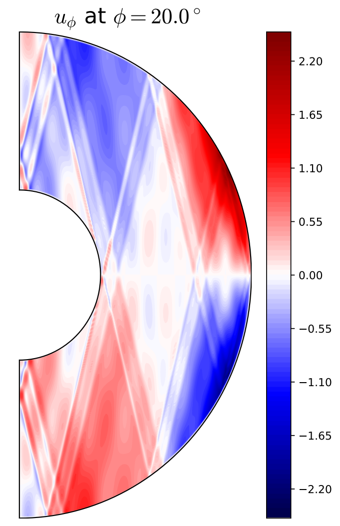


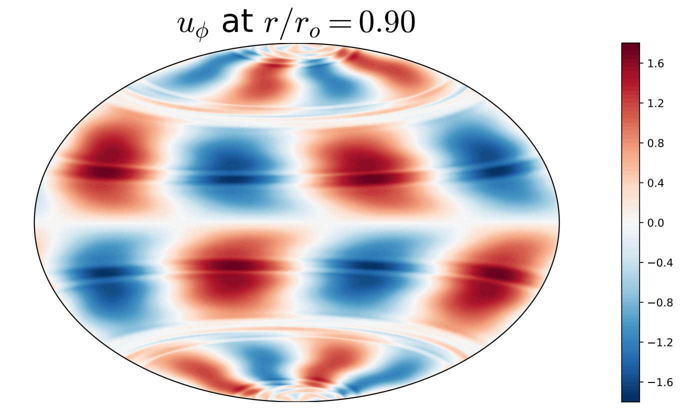
u_ϕ at $\phi = 0.0^\circ$
2.88
2.16
1.44
0.72
0.00
−0.72
−1.44
−2.16
−2.88



u_ϕ at $r/r_o = 0.90$
1.44
1.08
0.72
0.36
0.00
−0.36
−0.72
−1.08
−1.44

No slip : $(5,2)$


u_ϕ at $\phi = 20.0^\circ$
2.20
1.65
1.10
0.55
0.00
−0.55
−1.10
−1.65
−2.20


u_ϕ at $r/r_o = 0.90$
1.6
1.2
0.8
0.4
0.0
−0.4
−0.8
−1.2
−1.6

Slow vs fast modes

$$-\mathrm{i}\omega \boldsymbol{u} = -\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u}$$

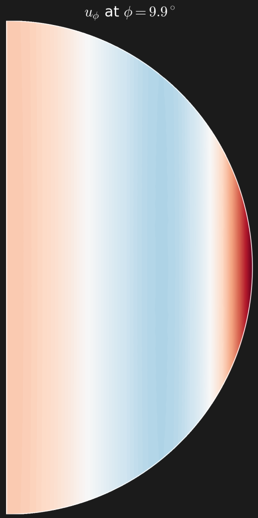
Slow vs fast modes

$$-\mathrm{i}\omega \boldsymbol{u} = -\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u}$$

$$\begin{aligned} |\omega| &\ll 1 \\ \Rightarrow |-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u}| &\ll 1 \end{aligned}$$

$\Rightarrow$ modes satisfy geostrophy to a large extent

“Rossby” modes



Live example

Code available here:

https://github.com/AnkitBarik/inermodz

Rossby waves

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- Occur in planets/stars

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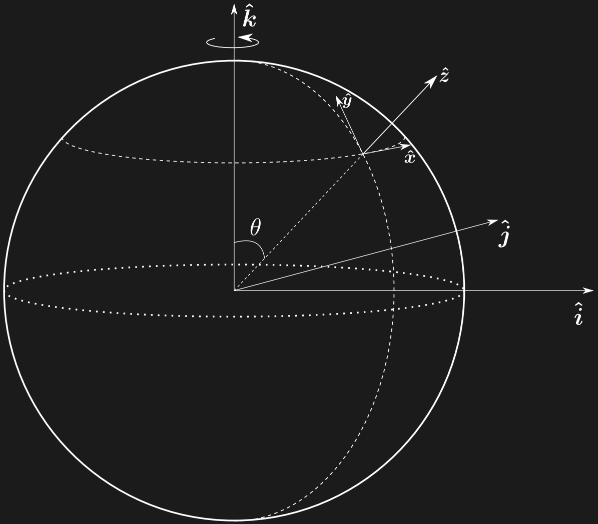
- Solve same equation as before, but set $u_r = 0$ from the beginning
- Two ways of doing this

Rossby waves : local way



$$
\begin{aligned}
&\boldsymbol{\Omega} \times \boldsymbol{u} \\
=&\boldsymbol{\Omega}\hat{\boldsymbol{k}} \times (u_x\hat{\boldsymbol{x}} + u_y\hat{\boldsymbol{y}}) \\
=&\Omega(\sout{\sin\theta\hat{\boldsymbol{y}}} + \cos\theta\hat{\boldsymbol{z}}) \\
&\times (u_x\hat{\boldsymbol{x}} + u_y\hat{\boldsymbol{y}})
\end{aligned}
$$

Rossby waves : local way

Coriolis force

$$
\begin{aligned}
&2\boldsymbol{\Omega} \times \boldsymbol{u} \\
=&2\Omega(y)(-u_y\hat{\boldsymbol{x}} + u_x\hat{\boldsymbol{y}})
\end{aligned}
$$

$$
\begin{aligned}
-\mathrm{i}\omega u_x - 2\Omega(y)u_y &= -\frac{\partial p}{\partial x} \\
-\mathrm{i}\omega u_y + 2\Omega(y)u_x &= -\frac{\partial p}{\partial y} \\
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} &= 0
\end{aligned} \tag{5}
$$

Rossby waves : local way

Assuming 2D plane waves and cross-differentiating,

$$\begin{aligned} k_y \omega u_x - 2\Omega(y) k_y u_y - 2u_y \frac{d\Omega}{dy} &= -\frac{\partial p}{\partial x \partial y} \\ k_x \omega u_y + 2\Omega(y) k_x u_x &= -\frac{\partial p}{\partial y \partial x} \\ k_x u_x + k_y u_y &= 0 \end{aligned} \tag{6}$$

Eliminating pressure,

$$\omega = -\frac{k_x}{k_x^2 + k_y^2} \frac{2d\Omega}{dy} \tag{7}$$

Rossby waves : local way

β-plane approximation: $\frac{2d\Omega}{dy} = \beta$

$$\omega = -\beta \frac{k_x}{k_x^2 + k_y^2} \tag{8}$$

Wave pattern always travels westward

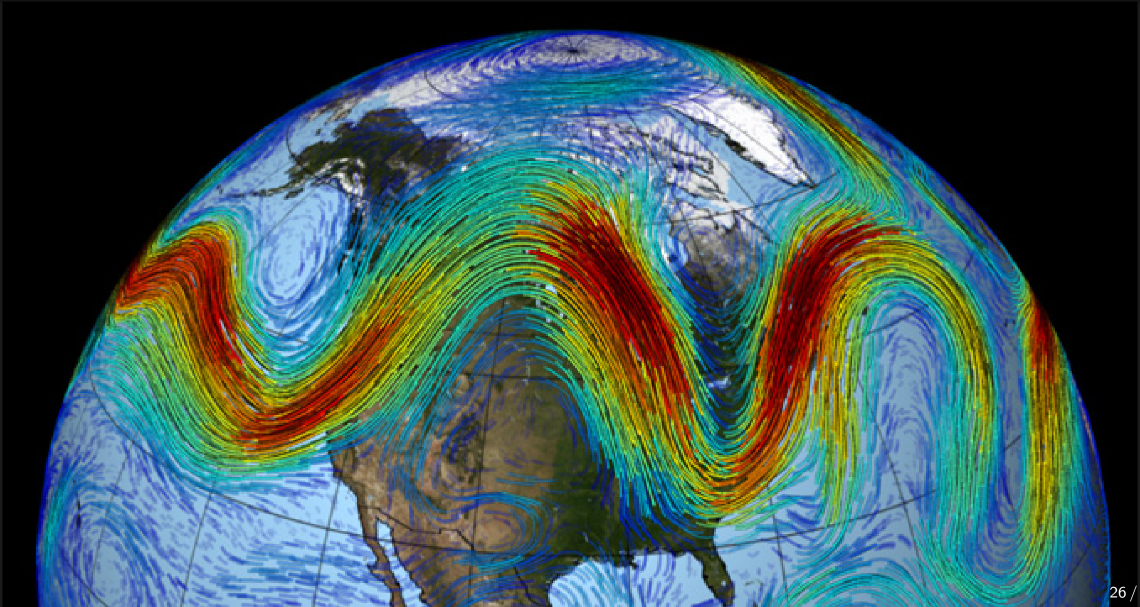
Rossby waves : local way

Group velocity

$$c_{\text{group}} = \beta \frac{k_x^2 - k_y^2}{k^4} \hat{\boldsymbol{x}} + 2\beta \frac{k_x k_y}{k^4} \hat{\boldsymbol{y}} \tag{9}$$

- When, $k_x^2 > k_y^2$ (*short waves*), energy propagates eastward
- When, $k_y^2 > k_x^2$ (*long waves*), energy propagates westward

Rossby waves



Rossby modes : global

Since $u_r = 0$,

$$\boldsymbol{u} = (\nabla \times \psi \hat{\boldsymbol{r}}) e^{-\mathrm{i}\omega t}$$

$$-\mathrm{i}\omega(\nabla \times \psi \hat{\boldsymbol{r}}) + 2\Omega \hat{\boldsymbol{z}} \times \nabla \times \psi \hat{\boldsymbol{r}} = -\nabla p$$

Applying $\hat{\boldsymbol{r}} \cdot \nabla \times$,

$$-\mathrm{i}\omega \nabla_H^2 \psi + 2\Omega \frac{\partial \psi}{\partial \phi} = 0$$

Rossby modes : global

$$-\mathrm{i}\omega\nabla_H^2\psi + 2\Omega\frac{\partial\psi}{\partial\phi} = 0$$

$$\psi = \sum \psi_{lm} Y_{lm}$$

$$\frac{\omega_{lm}}{\Omega} = -\frac{2m}{l(l+1)} \tag{10}$$

Back to inertial modes

Recall equation (4)

$$(l(\omega/2) + m)P_{lm}(\omega/2) = (l + m)P_{l-1,m}(\omega/2) \tag{11}$$

I can be proved that, when $l - m = 1$, $u_r = 0$ and

$$\frac{\omega_{lm}}{\Omega} = \frac{2}{m+1} \tag{12}$$

(Note that this (l, m) is independent of the ones used for Rossby waves earlier)

Back to inertial modes

Recall equation (4)

$$(l(\omega/2)+m)P_{lm}(\omega/2) = (l+m)P_{l-1,m}(\omega/2) \tag{11}$$

I can be proved that, when $l-m=1$, $u_r=0$ and

$$\frac{\omega_{lm}}{\Omega} = \frac{2}{m+1} \tag{12}$$

(Note that this (l,m) is independent of the ones used for Rossby waves earlier)
Some inertial modes are Rossby waves, but not all Rossby waves are inertial modes.

Rossby waves and inertial modes

Live demo

References

Zhang, K., Earnshaw, P., Liao, X., and Busse, F. H. (2001). On inertial waves in a rotating fluid sphere. *Journal of Fluid Mechanics*, 437(1):103–119.