# Boussinesq approximation

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## 1 Introduction

The 'Boussinesq approximation', sometimes called the 'Oberbeck-Boussinesq approximation' is a simplification of the Navier-Stokes and thermal energy equations often used to study phenomena such as thermal convection or internal gravity waves in fluids.

### Incompressible fluid

The first part of the approximation consists of considering the fluid as incompressible ( $\nabla \cdot \boldsymbol{u} = 0$ ). Let us see under what conditions this is valid. Starting from the continuity equation

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\
\Rightarrow\ &\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho (\nabla \cdot \boldsymbol{u}) = 0 \\
\Rightarrow\ &\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho\right) + \nabla \cdot \boldsymbol{u} = 0\,.
\end{aligned} \tag{1}
$$

The sound speed in a fluid is defined by

$$
c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s . \tag{2}
$$

Using this definition, we can write, $dp = c^2 d\rho$ and rewrite the above equation in terms of pressure:

$$
\frac{1}{\rho c^2}\left(\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \nabla p\right) + \nabla \cdot \boldsymbol{u} = 0\,. \tag{3}
$$

Let us non-dimensionalize this using a length scale $L$, a velocity scale $U$, a timescale $\tau = L/U$ and a density scale $\rho_0$ and a pressure scale, $P = \rho_0 U^2$. Representing the non-dimensional variables with $^*$, we get,

$$
\begin{aligned}
&\frac{1}{c^2 \rho_0 \rho^*}\left(\left(\frac{\rho_0 U^2 U}{L}\right)\frac{\partial p^*}{\partial t^*} + \left(\frac{\rho_0 U^2 U}{L}\right)\boldsymbol{u}^* \cdot \nabla p^*\right) + \left(\frac{U}{L}\right)\nabla \cdot \boldsymbol{u}^* = 0 \\
\Rightarrow &\frac{U^2}{c^2 \rho^*}\left(\frac{\partial p^*}{\partial t^*} + \boldsymbol{u}^* \cdot \nabla p^*\right) + \nabla \cdot \boldsymbol{u}^* = 0 \\
\Rightarrow &\frac{U^2}{c^2 \rho^*}\left(\frac{D p^*}{D t^*}\right) + \nabla \cdot \boldsymbol{u}^* = 0\,.
\end{aligned} \tag{4}
$$

The ratio of the characteristic fluid velocity to the sound speed, $U/c$ is called the Mach number $M$. Thus, the above equation can be written as,

$$M^2\left[\frac{1}{\rho^*}\left(\frac{Dp^*}{Dt^*}\right)\right]+\nabla\cdot\boldsymbol{u}^*=0\,. \tag{5}$$

Thus, the first term can only be ignored and the continuity equation can be reduced to $\nabla\cdot\boldsymbol{u}=0$, only when $M^2\ll 1$. Typically, $M<0.3$ is a good place for considering the fluid as an incompressible fluid. In planetary and stellar physics, the Mach number is often used to quantify compressibility of the flow. In compressible simulations, it is used to check how fast the velocities are and how fine the simulation grid should be.

For air, typical sound speed is 350 km/s, thus the above assumption is fairly good as long as our flow velocities are about 100 km/s. For liquid water, it is even easier to satisfy this, since the sound speed is 1470 km/s. For the outer core of the Earth, $c\sim$ 10km/s, and the flow speeds are only about $10^{-4}$ m/s .

Another situation where compressibility might be important can be found by looking at the hydrostatic equilibrium

$$\nabla p=\rho\boldsymbol{g}\,. \tag{6}$$

Using the definition of sound speed used above, we can write,

$$\begin{aligned}&c^2\nabla\rho=\rho\boldsymbol{g}\\ \Rightarrow\ &\frac{1}{\rho}\frac{d\rho}{dz}=-\frac{g}{c^2}\\ \Rightarrow\ &\rho(z)=\rho(z=0)e^{-z/(c^2/g)}\,.\end{aligned} \tag{7}$$

Thus, density variations happen over a height of $c^2/g$. This is called the density scale height. Thus, absolute incompressiblity can be assumed only when the height of our domain $H\ll c^2/g$, which is about 10 km for air. Is this assumption applicable to a lab experiment with liquid water?

### The momentum equation

Assuming an incompressible fluid (and a constant viscosity) simplifies the momentum equation to

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p+\rho\boldsymbol{g}+\mu\nabla^2\boldsymbol{u}\,. \tag{8}$$

In the absence of any flows, we obtain the hydrostatic equilibrium,

$$\nabla\overline{p}=\overline{\rho}\boldsymbol{g}\,. \tag{9}$$

where an overbar denotes an equilibrium value. Separating the density and pressure into equilibrium parts and perturbations around equilibrium (note that equilibrium velocity is zero),

$$\begin{aligned}\rho&=\overline{\rho}+\rho'\\ p&=\overline{p}+p'\,,\end{aligned} \tag{10}$$

we can re-write (8) as,

$$
\begin{aligned}
(\overline{\rho}+\rho')\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) &= -\nabla(\overline{p}+p')+(\overline{\rho}+\rho')\boldsymbol{g}+\mu\nabla^2\boldsymbol{u} \\
\Rightarrow (\overline{\rho}+\rho')\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) &= -\nabla p'+\rho'\boldsymbol{g}+\mu\nabla^2\boldsymbol{u} \\
\Rightarrow \left(1+\frac{\rho'}{\overline{\rho}}\right)\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) &= -\frac{1}{\overline{\rho}}\nabla p'+\frac{\rho'}{\overline{\rho}}\boldsymbol{g}+\nu\nabla^2\boldsymbol{u}\,,
\end{aligned} \tag{11}
$$

where in the second step, we have used the hydrostatic equilibrium : $-\nabla\overline{p}+\overline{\rho}\boldsymbol{g}=0$. For many incompressible fluids, the deviation from equlibrium is very small and thus, $\dfrac{\rho'}{\rho}\ll 1$. The Boussinesq approximation is that $\dfrac{\rho'}{\rho}\ll 1$ and can be ignored *except* for when it occurs in the buoyancy term. Both Oberbeck and Boussinesq realized that ignoring it everywhere would lead to no buoyancy at all. Using this approximation, we obtain,

$$
\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\overline{\rho}}\nabla p'+\frac{\rho'}{\overline{\rho}}\boldsymbol{g}+\nu\nabla^2\boldsymbol{u}\,. \tag{12}
$$

**The thermal energy equation**

The next part involves simplifying the thermal energy equation,

$$
\rho T\frac{Ds}{Dt} = \nabla\cdot(k\nabla T)+\Phi_\mu\,, \tag{13}
$$

where $s$ is entropy, $T$ is temperature, $k$ is thermal conductivity and $\Phi_\mu$ is viscous dissipation. We can make a further change of variables in terms of temperature $T$ and pressure $p$ using $ds=\left(\dfrac{\partial s}{\partial T}\right)_p dT+\left(\dfrac{\partial s}{\partial p}\right)_T dp = \dfrac{C_p}{T}dT-\dfrac{\alpha_T}{\rho}dp$, where $C_p$ is specific heat at constant pressure and $\alpha_T$ is the thermal expansion coefficiant.

We get,

$$
\rho C_p\frac{DT}{Dt}-\alpha_T T\frac{Dp}{Dt} = \nabla\cdot(k\nabla T)+\Phi_\mu\,. \tag{14}
$$

We make two further assumptions here:

1. Pressure variations are much smaller than temperature variations $\Rightarrow \rho C_p\dfrac{DT}{Dt}\gg\alpha_T T\dfrac{Dp}{Dt}$. This is true for incompressible liquids since $\alpha T\ll 1$.

2. Viscous dissipation has a negligible contribution to the thermal equation, $\Phi_\mu\ll\rho C_p\dfrac{DT}{Dt}$ .

The second assumption can be checked using,

$$
\frac{\Phi_\mu}{\rho C_p(DT/Dt)} \sim \frac{2\mu e_{ij}e_{ij}}{\rho\boldsymbol{u}\cdot\nabla T} \sim \frac{\nu}{C_p}\frac{U^2L^2}{\rho_0 U\delta T/L} = \frac{\nu}{C_p}\frac{U}{\delta T L}\,. \tag{15}
$$

This ratio must be $\ll 1$. Let's see what these ratios are in some cases. Table 1 provides typical parameters for ocean, atmosphere and the Earth's outer core. It is clear that for oceans and the outer core, it is an extremely good approximation. For atmospheres, it is still fairly okay, but much less so.

Often a third assumption is added, depending on the system : the thermal conductivity is constant, these assumptions simplify the thermal equation to

| | $\nu\,(\mathrm{m^2/s})$ | $C_p$ (J/kg/K) | $L$ (m) | $\rho\,(\mathrm{kg/m^3})$ | $\delta T$ (K) | $U$ (m/s) | Ratio |
|---|---|---|---|---|---|---|---|
| Ocean | $10^{-6}$ | 3850 | 3688 | $10^3$ | 30 | $5\times10^3$ | $10^{-11}$ |
| Atmosphere | $10^{-5}$ | 1 | $10^3$ | 1 | 10 | $10^5$ | $10^{-4}$ |
| Outer core | $10^{-6}$ | 850 | $2.265\times10^6$ | $10^4$ | $10^{-6}$ | $10^{-4}$ | $10^{-14}$ |

Table 1: Table of parameters to check the importance of viscous dissipation.

$$
\begin{aligned}
\rho C_p \frac{DT}{Dt} &= k\nabla^2 T \\
\Rightarrow \frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T &= \kappa\nabla^2 T\,,
\end{aligned}
\tag{16}
$$

where, $\kappa = k/(\rho C_p)$ is called the *thermal diffusivity* (units of $\mathrm{m^2/s}$).

We can separate out the temperature in an equilibrium part $\overline{T}$ and a deviation from equilibrium $T'$, $T = \overline{T} + T'$. For the equilibrium scenario, we have no flows and no time variation and we obtain,

$$
\nabla^2 \overline{T} = 0\,, \tag{17}
$$

which can be solved depending with boundary conditions (fixed temperature or fixed heat flux) to obtain $\overline{T}$. This variation is often assumed to be in a single direction, either $z$ in a Cartesian scenario, or $r$ while dealing with a spherical case. The equation then simplifies to either

$$
\frac{d^2\overline{T}}{dz^2} = 0\,, \tag{18}
$$

or,

$$
\frac{d}{dr}\left(r^2\frac{d\overline{T}}{dr}\right) = 0\,. \tag{19}
$$

In the presence of heat sources, the right hand side has a finite value $\epsilon$ instead of 0 and solution is different. We can now write equation (16) as,

$$
\frac{\partial}{\partial t}(\overline{T}+T') + \boldsymbol{u}\cdot\nabla(\overline{T}+T') = \kappa\nabla^2(\overline{T}+T')\,. \tag{20}
$$

Using $\partial\overline{T}/\partial t = 0$ and $\nabla^2\overline{T} = 0$, we obtain,

$$
\frac{\partial T'}{\partial t} + \boldsymbol{u}\cdot\nabla T' + \boldsymbol{u}\cdot\nabla\overline{T} = \kappa\nabla^2 T'\,. \tag{21}
$$

Since the variations in $\overline{T}$ are often assumed to be unidirectional, $\boldsymbol{u}\cdot\nabla\overline{T}$ often reduces to either $u_z\dfrac{d\overline{T}}{dz}$ or $u_r\dfrac{d\overline{T}}{dr}$.

### Using the equation of state

Using the thermal expansion coefficient $\alpha = -\dfrac{1}{\rho}\left(\dfrac{\partial\rho}{\partial T}\right)$, we can relate the density perturbations to the temperature perturbations as,

$$
\rho' = -\overline{\rho}\alpha T'\,. \tag{22}
$$

Substituting this in equation (12), we get the final form of the Navier-Stokes equation with the Boussinesq approximation,

$$
\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\overline{\rho}}\nabla p' - \alpha T'\boldsymbol{g} + \nu\nabla^2\boldsymbol{u}\,. \tag{23}
$$

**Final set of equations**

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\overline{\rho}} \nabla p' - \alpha T' \boldsymbol{g} + \nu \nabla^2 \boldsymbol{u} \tag{24}$$

$$\frac{\partial T'}{\partial t} + \boldsymbol{u} \cdot \nabla T' = -\boldsymbol{u} \cdot \nabla \overline{T} + \kappa \nabla^2 T' \tag{25}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{26}$$