Turbulence

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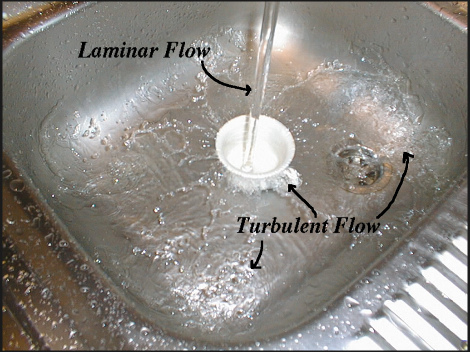
EPS Fluids

15 November 2023

Turbulence is ubiquitous



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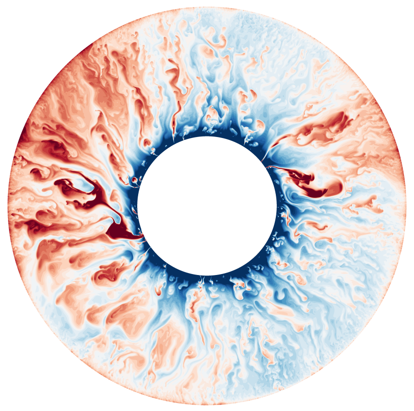

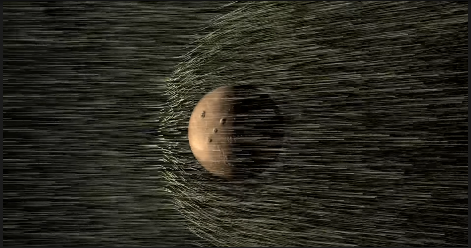

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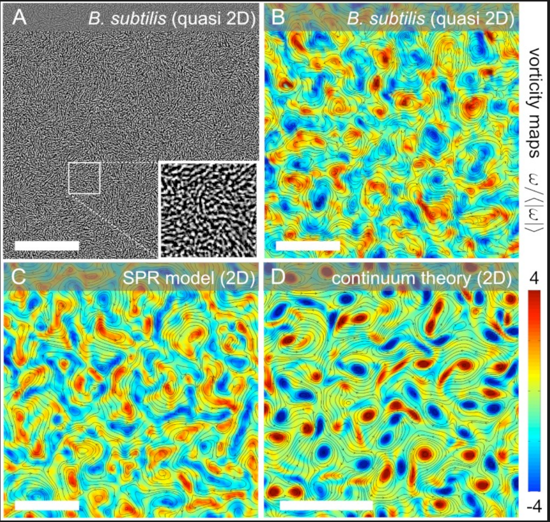
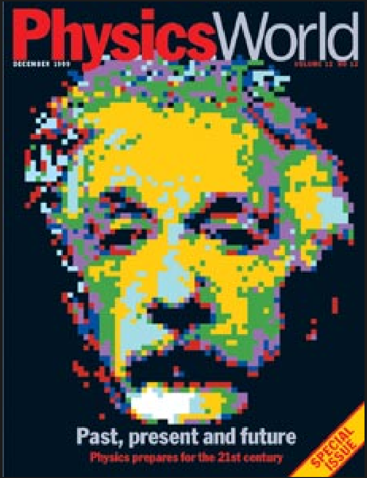


What is turbulence?



Physics World "Millennium Issue" - Ten great unsolved problems in physics:

1. Quantum gravity
2. Understanding the nucleus
3. Fusion energy
4. Climate change
5. Turbulence
6. Glassy materials
7. High-temperature superconductivity
8. Solar magnetism
9. Complexity
10. Consciousness

What is turbulence?

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- Non-linear
- Open systems
- Many degrees of freedom
- Highly irregular in space and time
- Often leads to a statistically quasi-stationary state far away from equilibrium
- Elements of chaos - butterfly effect

Navier-Stokes equation(s)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{Re} \nabla^2 \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

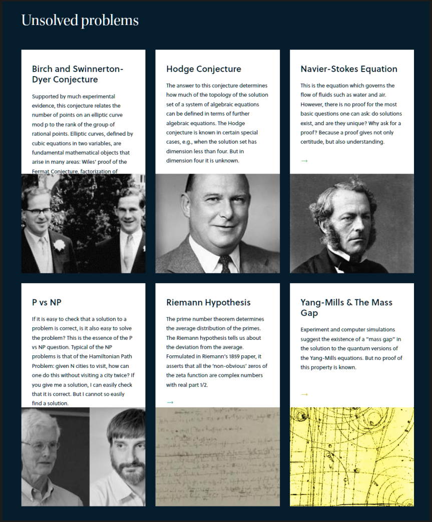
Reynolds number

$$Re = \frac{UL}{\nu}$$

- Importance of viscosity:
  - Low $Re \Rightarrow$ viscous laminar flow
  - High $Re \Rightarrow$ turbulent flow
- $Re > Re_c$ determines transition to turbulence
- Re_c depends on the system, $Re_c \approx 10000$ for pipe flows

$$\frac{|\boldsymbol{u} \cdot \nabla \boldsymbol{u}|}{|\nu \nabla^2 \boldsymbol{u}|} \sim \frac{U^2/L}{\nu U/L^2} = Re$$

Navier-Stokes equation(s)



Unsolved problems

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Navier-Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

→

P vs NP

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

→

Yang-Mills & The Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

→

Navier-Stokes equation(s)

EXISTENCE AND SMOOTHNESS OF THE NAVIER–STOKES EQUATION

CHARLES L. FEFFERMAN

The Euler and Navier–Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1 \le i \le n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \ge 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The *Navier–Stokes* equations are then given by

$$(1) \qquad \frac{\partial}{\partial t} u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n, t \ge 0),$$

$$(2) \qquad \operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n, t \ge 0)$$

Section 1

Analytical approach

Navier-Stokes equation(s)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla (p/\rho) + \nu \nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Non-locality

Take the divergence of Navier-Stokes and use $\nabla \cdot \boldsymbol{u} = 0$,

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Take the divergence of Navier-Stokes and use $\nabla \cdot \boldsymbol{u} = 0$,

$$\nabla^2 (p/\rho) = -\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})$$

Inverting for p/ρ gives,

$$p(\boldsymbol{r})/\rho = \frac{1}{4\pi} \int \frac{\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3\boldsymbol{r}'$$

where we have used the Green's function solution to Poisson's equation

Non-locality

Pressure depends on the whole velocity field over the domain, which in turn determines the velocity field.

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla \left[\frac{1}{4\pi} \int \frac{\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3\boldsymbol{r}' \right] + \nu \nabla^2 \boldsymbol{u}$$

Non-linear (because of $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$) as well as non-local (because of the integral).

Section 2

Statistical approach

Statistical approach

Average over ensembles (several realisations of the same event) or time

Closure problem

Closure problem

Hierarchy of equations of the form:

$$\frac{\partial}{\partial t}(\text{statistical property of } \boldsymbol{u}) = \text{Function }(\text{other properties of } \boldsymbol{u})$$

Always more number of unknowns than equations.

Reynolds Averaged Navier Stokes (RANS)

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \sum_j \frac{\partial^2 u_i}{\partial x_j^2}$$

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Split into mean and fluctuating parts:

$$u_i = \langle u_i \rangle + u_i'$$
$$p = \langle p \rangle + p'$$

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Reynolds Averaged Navier Stokes (RANS)

Following Reynolds rules (Reynolds, 1883),

$$
\begin{gathered}
\langle u' \rangle = 0; \;\; \langle c \langle u \rangle \rangle = c \langle u \rangle \\
\langle u + v \rangle = \langle u \rangle + \langle v \rangle \\
\langle \langle u \rangle v \rangle = \langle u \rangle \langle v \rangle \\
\left\langle \frac{\partial u}{\partial t} \right\rangle = \frac{\partial \langle u \rangle}{\partial t}
\end{gathered}
$$

Reynolds Averaged Navier Stokes (RANS)

Obtain equation for mean flow

$$\frac{\partial \langle u_i \rangle}{\partial t} + \sum_j \left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle = -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} + \nu \sum_j \frac{\partial^2 \langle u_i \rangle}{\partial x_j^2} - \sum_j \frac{\partial}{\partial x_j} \langle u_i' u_j' \rangle$$

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Closure problem

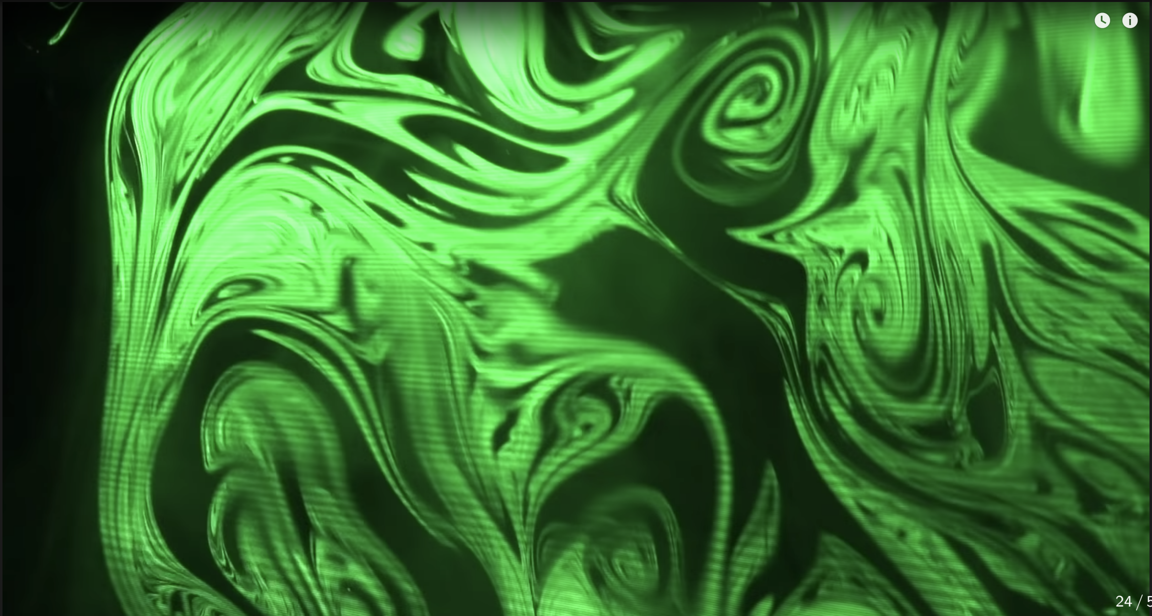
$$\frac{\partial}{\partial t}\langle u_i' u_j' \rangle = \text{Term containing } u_i' u_j' u_k' + \ldots$$

and so on.
Must close the system with some model.

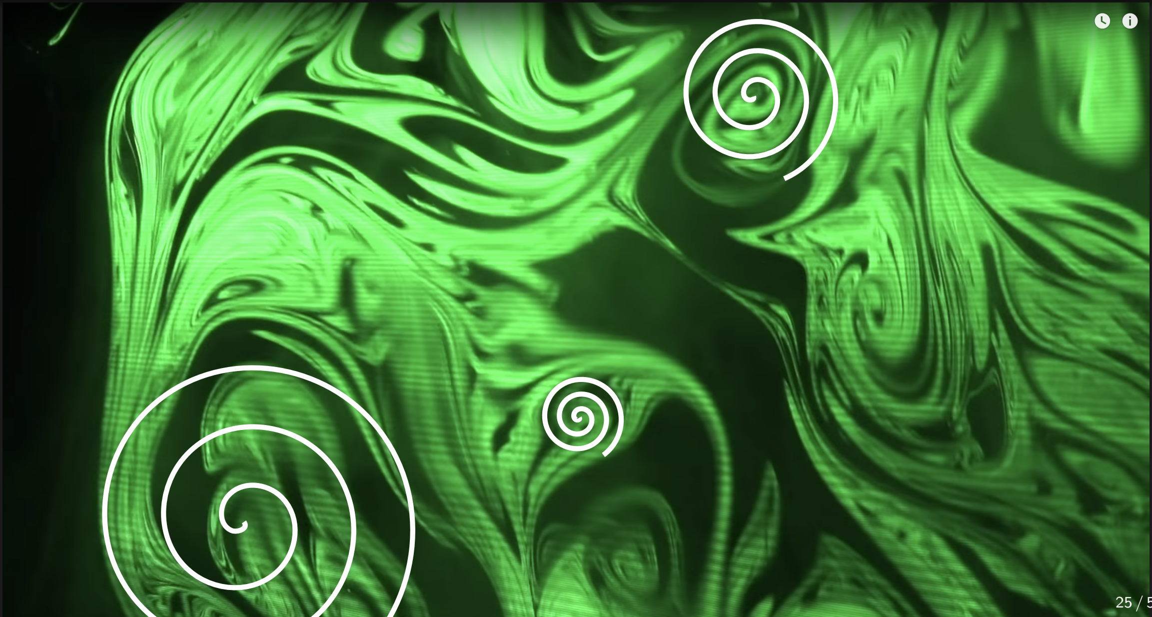
Eddy viscosity

The first closure model attempt was by Joseph Valentin Boussinesq $\rightarrow$ eddy viscosity.

Eddies



Eddies



Eddy viscosity

$$-\langle u_i' u_j' \rangle = \nu_t \frac{\partial \langle u_i \rangle}{\partial x_j}$$

Navier-Stokes:

$$\frac{\partial \langle u_i \rangle}{\partial t} + \sum_j \left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle = -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} + \nu \sum_j \frac{\partial^2 \langle u_i \rangle}{\partial x_j^2} - \sum_j \frac{\partial}{\partial x_j} \langle u_i' u_j' \rangle$$

becomes,

$$\frac{\partial \langle u_i \rangle}{\partial t} + \sum_j \left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle = -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} + (\nu + \nu_t) \sum_j \frac{\partial^2 \langle u_i \rangle}{\partial x_j^2}$$

Turbulent dissipation

As ν decreases, the eddies get smaller and gradients in velocity get larger, adding to enhanced dissipation. Thus, dissipation in the low viscosity limit stays finite.

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Perform $\boldsymbol{u}\cdot$ (Navier-Stokes) and integrate over volume.

$$\frac{d}{dt}\int \frac{1}{2}|\boldsymbol{u}|^2 dV = -\nu \int |\boldsymbol{\omega}|^2 dV$$

where, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ =vorticity.

Zeroth law

Zeroth law states that:

$$\lim_{\nu \to 0} \nu \int |\boldsymbol{\omega}|^2 dV \neq 0$$

- As $\nu \to 0$ or $Re \to \infty$, dissipation becomes independent of viscosity
- Euler (no-viscosity) and the Navier-Stokes equations (with viscosity) are fundamentally physically different

Turbulent diffusion

- Turbulence creates small scale motions and enhances diffusion
- Think of a cookie batter - you can either stir it (turbulent diffusion), or let it diffuse and mix by itself (molecular diffusion)
- Math similar to the previous analysis of turbulent or eddy viscosity

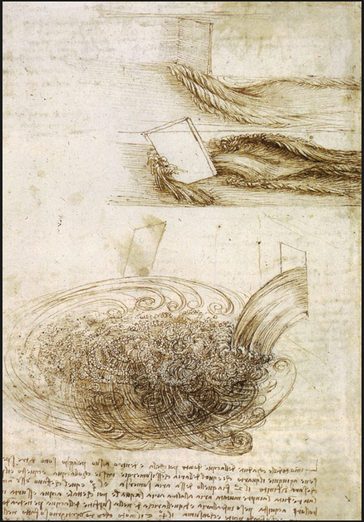
Turbulence - necessary evil

- Absolutely necessary to enhance diffusion in everyday life
- Necessary for astrophysical processes (e.g.: transport of angular momentum, generation of magnetic fields)
- Dissipates energy and produces heat in systems
- Produces some other challenges:
  - Design of airplanes, cars, ships etc.
  - Weather prediction
  - Fusion energy

Section 3

Enter Kolmogorov

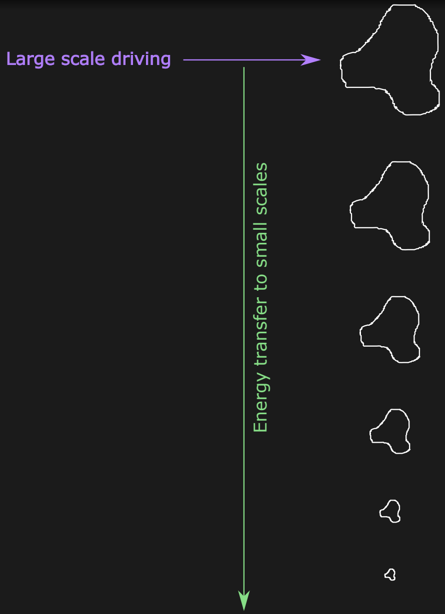
Length scales



Length scales

- Turbulence consists of a whole continuum of length scales (l) or wavenumbers ($k = 2\pi/l$)
- The system is driven at a large scale (L)
- Dissipation is strongest at the smallest scale (η)
- Energy in $=$ Energy out
- How does energy transport occur?

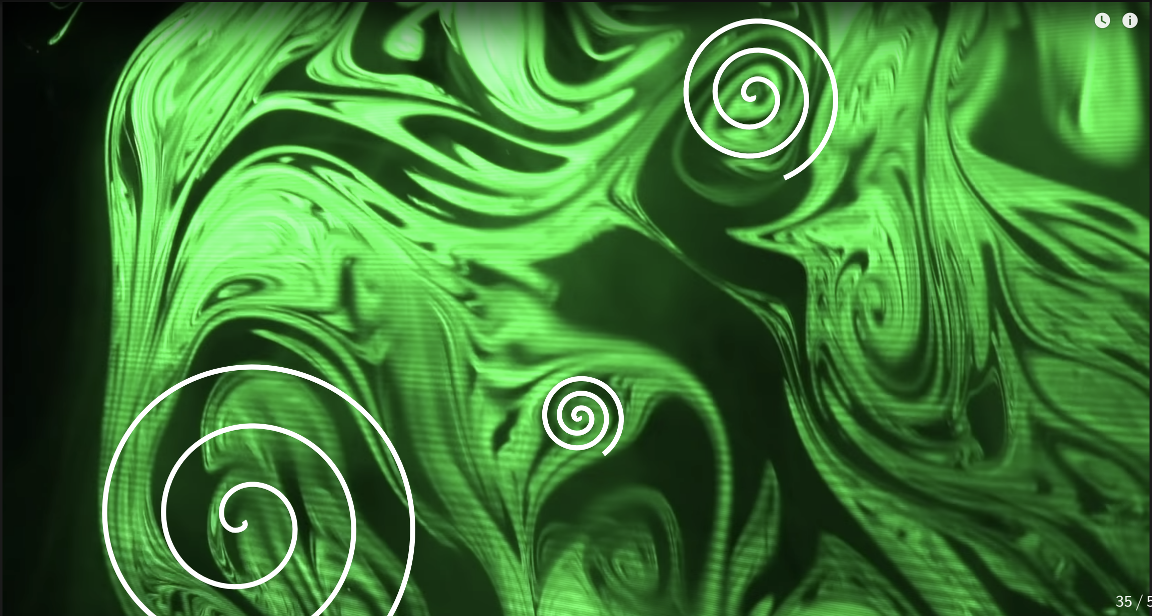
The Richardson cascade



"Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity."
- Lewis Fry Richardson

(rewording Jonathan Swift: *"Great fleas have little fleas upon their backs to bite 'em, And little fleas have lesser fleas, and so ad infinitum."*)

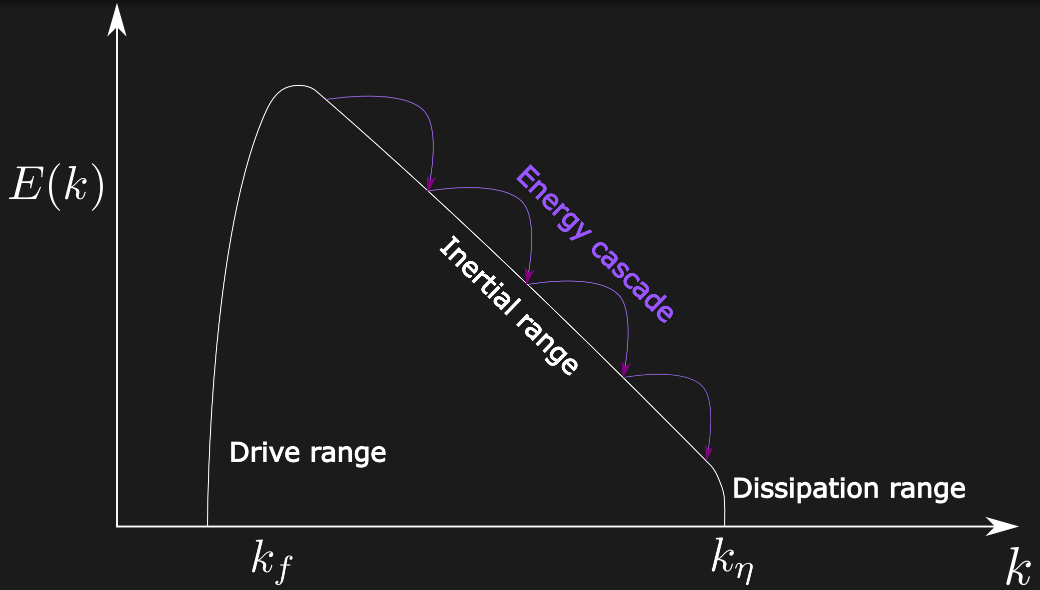
The Richardson cascade



The Richardson cascade

- Flux of energy from largest scale (L) to smallest scale (η) is a multi-step process
- Energy is first passed from a scale l_0 to a smaller scale l_1 to an even smaller scale l_2 and so on, through inertia (that $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ thingy)
- The above goes on till the smallest scale (η) is reached when viscosity becomes very important
- Energy transport is "local" in length-scales or wavenumbers

The Richardson cascade



Kolmogorov microscales

- The zeroth law implies in the limit $Re \to \infty$, rate of dissipation of kinetic energy per unit mass, ϵ is independent of viscosity and also, the smallest scales where the dissipation takes place.
- Since energy "injected" into large scales must equal energy dissipated, ϵ is purely a function of U and L (also called the "integral scales" or "injection scales").
- ϵ has units of (energy/mass/time) $= \mathrm{m}^2/\mathrm{s}^3$, dimensional analysis gives us,

$$\epsilon \sim \frac{U^3}{L}$$

Kolmogorov microscales

- The scale at which dissipation takes place (η, unit m) depends on rate of dissipation (units m^2/s^3) and viscosity (units m^2/s). Dimensional analysis gives us,

$$\eta \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} = Re^{-3/4}L$$

- Similar analysis also gives us the velocity at the smallest scale,

$$u_\eta \sim (\nu\epsilon)^{1/4} = Re^{-1/4}U$$

- These are known as the Kolmogorov microscales after A. N. Kolmogorov.

Kolmogorov microscales

- Note that $\frac{u_\eta \eta}{\nu} \sim 1 \Rightarrow$ viscosity is equally important as inertia
- Ratio between smallest and largest scales,

$$\frac{\eta}{L} \sim Re^{-3/4}$$

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- In addition, keeping up with the length scale, the time resolution also varies as $Re^{3/4}$
- Thus, total computational cost increases as $Re^{9/4}Re^{3/4} = Re^3$, limiting simulations to not too high Reynolds numbers.

The Kolmogorov 2/3 law

- Flux of energy is constant through the cascade, thus eddies of any length scale (l) must satisfy

$$\epsilon \sim \frac{u_l^3}{l}$$

  assuming all eddies are equally "space-filling".

- Can be re-written as

$$u_l^2 \sim (\epsilon l)^{2/3}$$

The Kolmogorov-Obukhov 5/3 law

(Kolmogorov, 1941a,b; Obukhov, 1941)

- Consider energy $\mathcal{E}(k)$ carried by flow between wavenumbers k and $k + dk$

$$\int_0^\infty \mathcal{E}(k)dk = E = \frac{1}{2V}\int |\boldsymbol{u}|^2 dV$$

$$\mathcal{E}(k) = \frac{\partial E}{\partial k}$$

- Units of $\mathcal{E}(k) = \text{m}^2/\text{s}^2/(1/\text{m}) = \text{m}^3/\text{s}^2$

The Kolmogorov-Obukhov 5/3 law

Assumptions:

- Homogeneous and isotropic turbulence
- Scale invariance ⇒ " there exists a range of scales (the inertial range) in which effects of viscosity, boundary conditions, and large-scale structures are not important" (Meneveau and Katz, 2000)

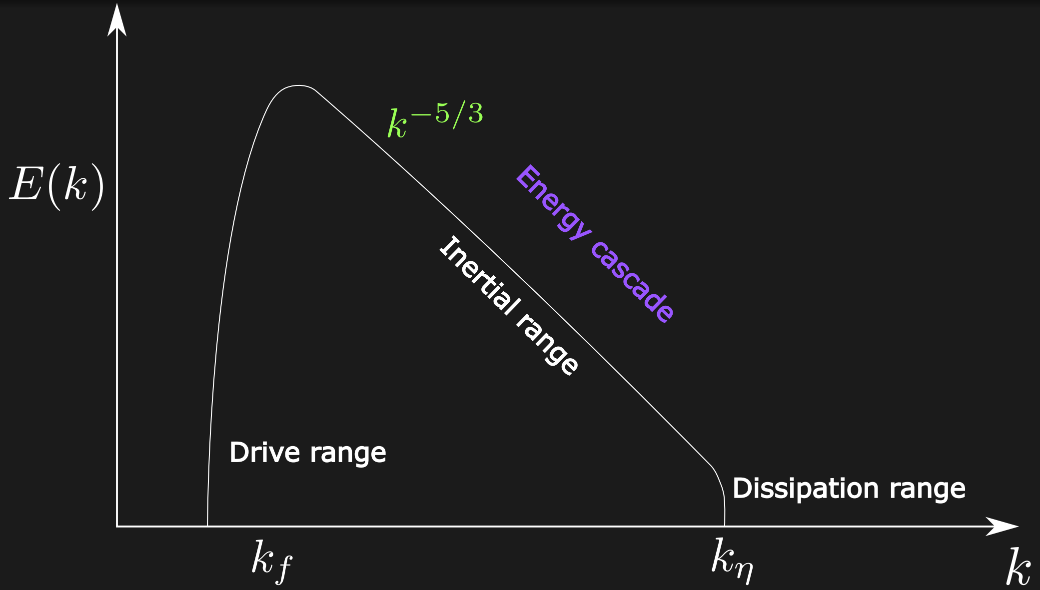
The Kolmogorov-Obukhov 5/3 law

$\mathcal{E}(k)$ (units m^3/s^2) is a function of ϵ (units m^2/s^3) and k (unit 1/m). Dimensional analysis gives,

$$\mathcal{E}(k) = \alpha \epsilon^{2/3} k^{-5/3}$$

This is the most famous result in turbulence, called the "5/3" law, sometimes also called "K41 theory".

The Kolmogorov-Obukhov 5/3 law



2D Turbulence

- Important for geophysical/planetary fluid dynamics
- Leads to both forward and inverse cascades

2D Turbulence

Invariant quantities:

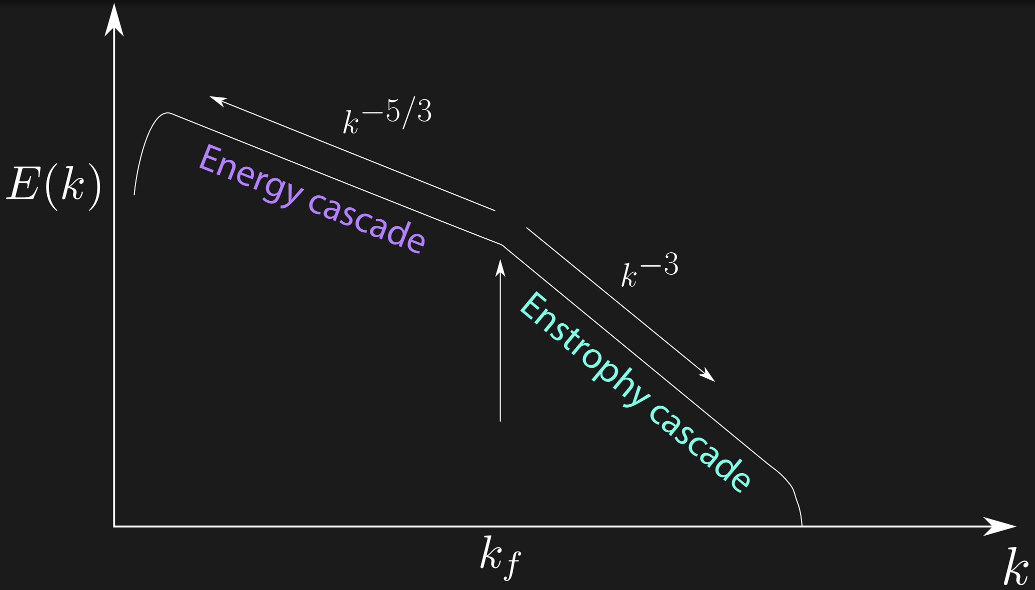
$$\text{Energy: } \mathcal{E} = \frac{1}{2}\int_v |\boldsymbol{u}|^2 dv$$
$$\text{Enstrophy: } \Omega = \int_v |\boldsymbol{\omega}|^2 dv$$

Cascades in 2D turbulence : triadic interactions and resonances

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \dots$$

- If the LHS has a wavenumber k_3, $e^{ik_3 x}$ and the RHS has two wavenumbers k_1 and k_2, then interactions such that $k_3 = k_1 \pm k_2$ will feed energy into k_3.
- If $\boldsymbol{u}$ consists of waves (say Rossby waves) of the form $e^{i(kx-\omega t)}$, then a resonant excitation can occur if $k_3 = k_1 \pm k_2$ and $\omega_3 = \omega_1 \pm \omega_2$.

Cascades in 2D turbulence



Challenges and open questions

- Effects of inhomogeneity, anisotropy, compressiblity
- Derivation 5/3 law from Navier-Stokes
- First principle simulations of complex problems not feasible
- Often, one is interested in only large scales - how do we get rid of small scales in simulations? e.g.: Large Eddy Simulations (LES) + sub-grid scale models.

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Acknowledgments and further reading

- Books
  - *Turbulence in rotating, stratified and electrically conducting fluids* - P. A. Davidson
  - *A First Course in Turbulence* - Tennekes and Lumley
  - *Turbulence: The Legacy of A. N. Kolmogorov* - Uriel Frisch
  - *Turbulent Flows* - Stephen B. Pope
- Other stuff
  - 3Blue1Brown video: https://youtu.be/_UoTTq651dE
  - Presentation by Frank Jenko for Les Houches winter school: http://www.ens-lyon.fr/PHYSIQUE/Equipe2/LesHouches15/Talks_files/Jenko-1.pdf
  - Wiki on RANS: https://en.wikipedia.org/wiki/Reynolds-averaged_Navier%E2%80%93Stokes_equations
  - Wiki on Turbulence modelling: https://en.wikipedia.org/wiki/Turbulence_modeling
  - Notes on Kolmogorov microscales: https://my.eng.utah.edu/~mcmurtry/Turbulence/turblt.pdf
  - Notes on turbulence: https://www.uio.no/studier/emner/matnat/math/MEK4300/v13/undervisningsmateriale/tb_16january2013.pdf
  - Notes on turbulence: https://www.krellinst.org/doecsgf/conf/2011/pres/moin_notes.pdf

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- Wingtip vortex: NASA Langley Research Center (https://en.wikipedia.org/wiki/Wingtip_vortices#/media/File:Airplane_vortex_edit.jpg)
- Convection simulation: Nathanaël Schaeffer (https://figshare.com/articles/Temperature_field_in_the_equatorial_plane_of_the_Earth_s_core_from_a_high_resolution_numerical_simulation/3502370)
- Solar wind on Mars: NASA Maven (https://youtu.be/d0lljDQURgo)
- Horsehead nebula: NASA JPL (https://www.jpl.nasa.gov/spaceimages/index.php?search=horsehead)
- Bacterial suspension figure: Wensink et al. (2012)
- Laser visualisation: 3Blue1Brown (https://youtu.be/_UoTTq651dE)