Laminar flow

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Recap : non-dimensionalized Navier-Stokes

$$\frac{L}{UT}\frac{\partial \boldsymbol{u}_*}{\partial t_*} + \boldsymbol{u}_* \cdot \nabla \boldsymbol{u}_* = -\nabla p_* + \frac{1}{Fr}\hat{\boldsymbol{z}} + \frac{1}{Ro}\hat{\boldsymbol{\Omega}} \times \boldsymbol{u}_* + \frac{1}{Re}\nabla^2 \boldsymbol{u}_*$$

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- When $Re \ll 1$, viscous forces dominate and the flow is 'laminar'
- We consider flows where rotation can be neglected $(Ro \gg 1)$
- The absence of $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ term allows for analytical solutions

“Dynamic” pressure

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Subtracting,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p_d + \nu \nabla^2 \boldsymbol{u}$$

$p_d \rightarrow$ dynamic pressure

What do you need for laminar flow?

$$Re = \frac{UL}{\nu}$$

Laminar flows

- Parallel streamlines
- Time reversible
- Looks “frozen in time” or glass-like appearance

Some cool videos

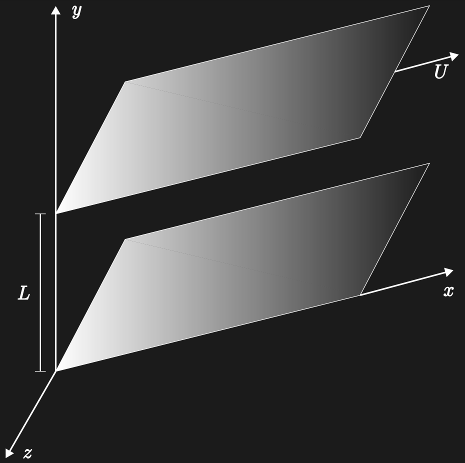
Video 1 : https://youtu.be/K9coK-Le6i0?t=247

Video 2 : https://youtu.be/57IMufyoCnQ?t=202

Useful?

Cleanrooms!

Parallel plates



Driven by top plate moving at velocity U in x-direction and a pressure gradient ∇p

Parallel plates

"Fully developed flow" $\rightarrow$ Far away from edges and steady-state $\dfrac{\partial}{\partial t} = 0$

Parallel plates

2D nature of the problem dictates : $\dfrac{\partial}{\partial z} = 0$

Mass continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

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Thus, $\dfrac{\partial v}{\partial y} = 0$

Since $v = 0$ at $y = 0$, $v = 0$ everywhere.

Parallel plates

Navier-Stokes get reduced to:

$$\begin{aligned} 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dy^2} \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned} \tag{1}$$

Second equation implies $p \equiv p(x)$.

Parallel plates

First equation:

$$
\begin{aligned}
0 &= -\frac{dp}{dx} + \rho\nu\frac{d^2u}{dy^2} \\
\mu\frac{d^2u}{dy^2} &= \frac{dp}{dx}
\end{aligned} \tag{2}
$$

Parallel plates

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\begin{aligned}
0 &= -\frac{dp}{dx} + \rho\nu\frac{d^2u}{dy^2} \\
\mu\frac{d^2u}{dy^2} &= \frac{dp}{dx}
\end{aligned} \tag{2}
$$

Integrating twice,

$$\mu u = \frac{dp}{dx}\frac{y^2}{2} + Ay + B \tag{3}$$

Parallel plates

$$\mu u = \frac{dp}{dx}\frac{y^2}{2} + Ay + B$$

$u(y = 0) = 0 \Rightarrow B = 0$

$u(y = L) = U \Rightarrow A = \frac{1}{L}\left(\mu U - \frac{dp}{dx}\frac{L^2}{2}\right) = \frac{\mu U}{L} - \frac{dp}{dx}\frac{L}{2}$

We get,

$$u = \frac{U}{L}y + \frac{y}{2}\frac{dp}{dx}(y - L) \tag{4}$$

Special cases

Plane Couette flow: $\frac{dp}{dx} = 0$

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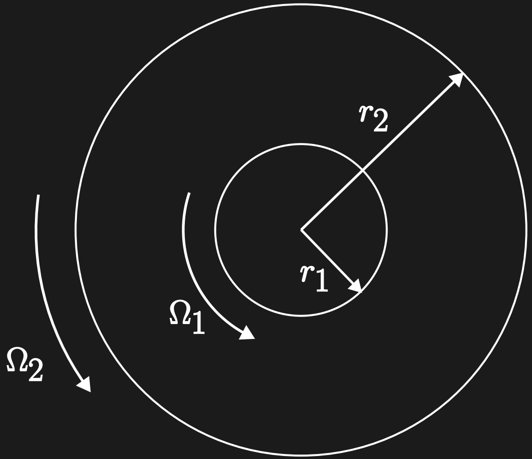
Plane Poiseuille flow: $U = 0$

$$u = \frac{y}{2} \frac{dp}{dx} (y - L) \tag{6}$$

Flow vectors

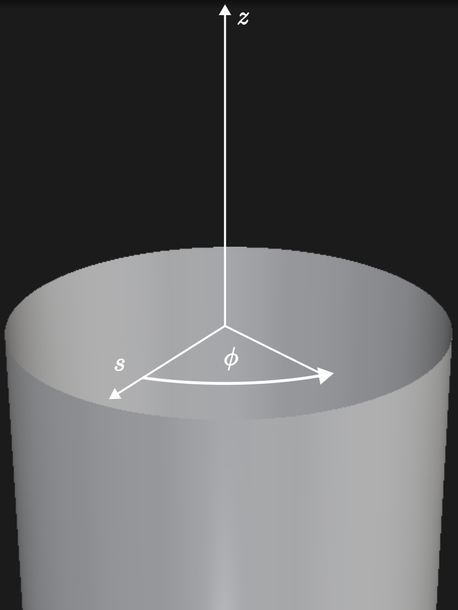
How does the flow look like?

Taylor-Couette flow



Also called Circular Couette flow or Cylindrical Couette flow

Taylor-Couette flow



Cylindrical coordinate system: (s, ϕ, z)

s : along cylindrical radius

ϕ : azimuth

z : along the axis of the cylinder

Symmetry

$$\frac{\partial}{\partial \phi} = 0 \text{ and } \frac{\partial}{\partial z} = 0$$

Mass conservation

$$\frac{1}{s}\frac{d}{ds}(su_s) = 0 \Rightarrow su_s = \text{constant} \tag{7}$$

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$u_s(s = r_1) = 0 \Rightarrow u_s = 0$ everywhere.

Equations of motion

s-direction:

$$-\frac{u_\phi^2}{s} = -\frac{1}{\rho}\frac{dp}{ds} \tag{8}$$

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Pressure increases radially outward because of the centrifugal force.

Equations of motion

ϕ-direction:

$$\begin{aligned} \mu(\nabla^2 \boldsymbol{u})_\phi &= 0 \\ \Rightarrow \mu\left[\frac{1}{s}\frac{d}{ds}\left(s\frac{d}{ds}u_\phi\right) - \frac{u_\phi}{s^2}\right] &= 0 \\ \Rightarrow \frac{d^2 u_\phi}{ds^2} + \frac{1}{s}\frac{du_\phi}{ds} - \frac{u_\phi}{s^2} &= 0 \end{aligned} \tag{9}$$

Equations of motion

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Assume an ansatz $u_\phi = s^m$, obtain the indicial equation:

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

Equations of motion

Thus,

$$u_\phi = As + B/s \tag{10}$$

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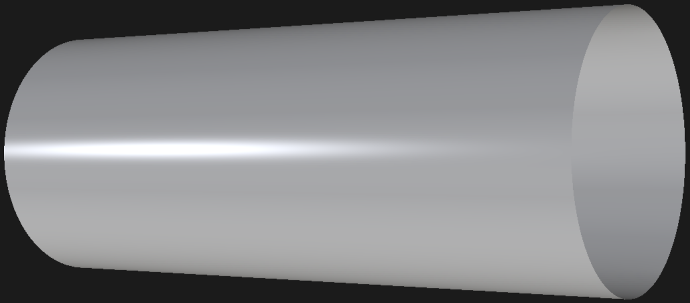
Using boundary conditions, $u_\phi(r_1) = \Omega_1 r_1$ and $u_\phi(r_2) = \Omega_2 r_2$,

$$\begin{aligned} A &= \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2} \\ B &= \frac{(\Omega_1 - \Omega_2) r_1^2 r_2^2}{r_2^2 - r_1^2} \end{aligned} \tag{11}$$

Streamlines

How to the flow streamlines look like?

Pipe flow



- Flow through a pipe along its axis.
- Using cylindrical coordinates again, x along the pipe plays the role of z now.
- Only non-zero component of velocity is in the x-direction.

Equations of motion

s-direction:

$$0 = -\frac{\partial p}{\partial s} \tag{12}$$

Equations of motion

s-direction:

$$0 = -\frac{\partial p}{\partial s} \tag{12}$$

Pressure is a function of x alone.

Equations of motion

x-direction:

$$0 = -\frac{dp}{dx} + \frac{\mu}{s}\frac{d}{ds}\left(s\frac{du_x}{ds}\right) \tag{13}$$

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First term is a function of x, second is a function of $s \Rightarrow$ both must be constant $\rightarrow$ pressure varies linearly along the pipe.

Equations of motion

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Integrating twice, we get,

$$u_x = \frac{s^2}{4\mu}\frac{dp}{dx} + A\ln s + B \qquad (14)$$

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Integrating twice, we get,

$$u_x = \frac{s^2}{4\mu}\frac{dp}{dx} + A\ln s + B \tag{14}$$

Since u_x has to be finite at the axis, $A = 0$. Using $u_x = 0$ at $s = a$, where a is the radius of the pipe, we get, $B = -\frac{a^2}{4\mu}\frac{dp}{dx}$.

Equations of motion

$$u_x = \frac{s^2 - a^2}{4\mu}\frac{dp}{dx} \tag{15}$$

Flow vectors

How do the flow vectors look like?