Gravity Waves

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EPS Fluids

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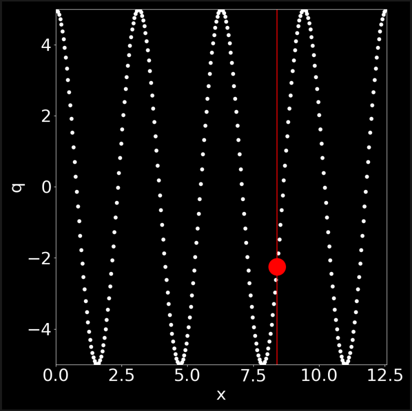
Waves



- Transport energy
- No physical movement of matter
- Transport of fluctuations of different quantities (e.g.: pressure, density, velocity, magnetic field, current density etc.)

The wave equation in 1D

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

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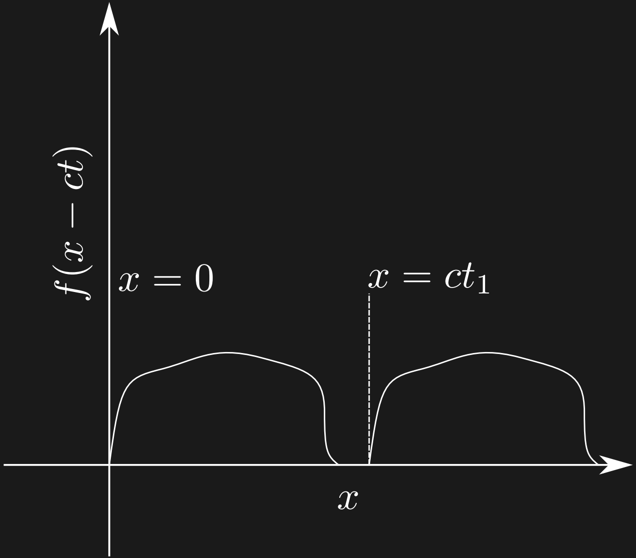
Solution:

$$\begin{aligned} &\xi = x - ct, \eta = x + ct \\ &\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \\ &\frac{\partial u}{\partial \xi} = 0 \text{ or } \frac{\partial u}{\partial \eta} = 0 \\ &u(x,t) = f(\xi) + g(\eta) = f(x - ct) + g(x + ct) \end{aligned} \tag{2}$$

What does it mean?



c is the speed of propagation, also known as "phase speed"

Sinusoids

- Any function f can be decomposed into a sum of sinusoidal functions
- We focus on a sinusoidal wave:

$$u(x,t) = a \sin\left[\frac{2\pi}{\lambda}(x - ct)\right] \tag{3}$$

Important quantities

Wavenumber

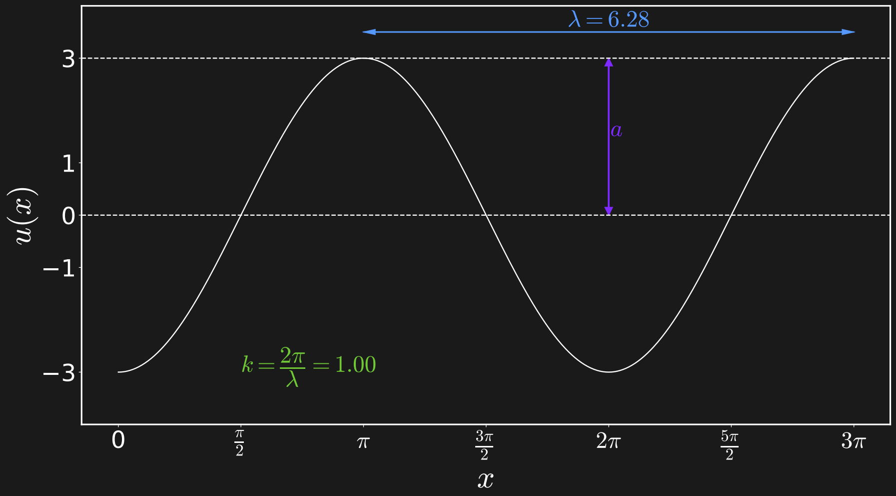
$$u(x,t) = a \sin\left[\boxed{\frac{2\pi}{\lambda}}(x - ct)\right]$$

Phase speed

Amplitude

Wavelength

Important quantities



Alternative forms

$$u = a \sin\left[\frac{2\pi}{\lambda}(x - ct)\right] = a \sin[k(x - ct)] = a \sin(kx - \omega t) \qquad (4)$$

- Period: $T = \dfrac{\lambda}{c}$
- Frequency: $\nu = \dfrac{1}{T} = \dfrac{c}{\lambda}$
- Circular or angular frequency: $\omega = 2\pi\nu = ck$
- $c = \dfrac{\omega}{k} =$ phase speed

In 3D

$$\nabla^2 \boldsymbol{u} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = 0 \tag{5}$$

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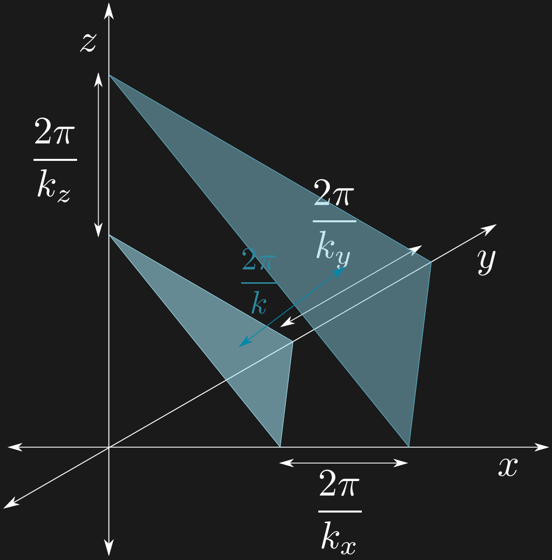
$$\boldsymbol{u} = a\sin(k_x x + k_y y + k_z z - \omega t) = a\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) \tag{6}$$

- Wavevector: $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}$
- Position vector: $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$
- Wavelength: $\lambda = \dfrac{2\pi}{|\boldsymbol{k}|} = \dfrac{2\pi}{\sqrt{k_x^2 + k_y^2 + k_z^2}}$
- Phase speed: $\boldsymbol{c} = \dfrac{\omega}{k}\hat{\boldsymbol{k}}$

In 3D



Complex form

Using the Euler identity: $Ae^{\mathrm{i}\theta} = A(\cos\theta + \mathrm{i}\sin\theta)$,

In 1D:

$$u = ae^{\mathrm{i}(kx - \omega t)} \tag{7}$$

In 3D:

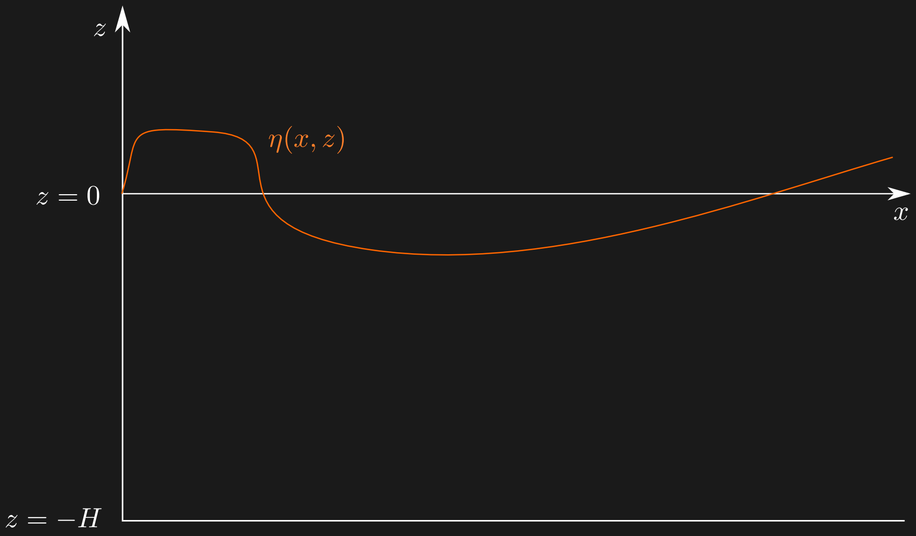
$$\boldsymbol{u} = ae^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \tag{8}$$

- Easier to treat mathematically
- Only real part contributes to solution

Types of waves in planetary fluid dynamics

Type of wave	Restoring force(s)
Acoustic (p-modes)	$-\nabla p$
Inertial	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u}$
Surface/Internal gravity (f-/g-modes)	$\rho' \boldsymbol{g}$
Inertia-gravity or gravito-inertial	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} + \rho' \boldsymbol{g}$
Alfvén	$\boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magnetoacoustic	$-\nabla p + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magneto-Coriolis (MC)	$-\nabla p - 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$
Magnetic, Archimedean, Coriolis (MAC)	$-2\boldsymbol{\Omega} \times \boldsymbol{u} + \rho' \boldsymbol{g} + \boldsymbol{B} \cdot \nabla \boldsymbol{B}$

Surface gravity waves



Surface gravity waves

Assumptions:

- Inviscid fluid
- Velocity of fluid, $\boldsymbol{u}$ is irrotational.

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$$\begin{gathered}\nabla \times \boldsymbol{u} = 0 \Rightarrow \boldsymbol{u} = \nabla\phi \\ u_x = \frac{\partial \phi}{\partial x},\ u_z = \frac{\partial \phi}{\partial z}\end{gathered} \tag{9}$$

Surface gravity waves

Assumptions:

- Inviscid fluid
- Velocity of fluid, $\boldsymbol{u}$ is irrotational.

$$\nabla \times \boldsymbol{u} = 0 \Rightarrow \boldsymbol{u} = \nabla\phi$$
$$u_x = \frac{\partial\phi}{\partial x},\ u_z = \frac{\partial\phi}{\partial z} \tag{9}$$

Continuity equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{10}$$

Boundary conditions

$$u_z = \frac{\partial \phi}{\partial z} = 0, \;\; z = -H \tag{11}$$

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$$\frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta = u_z(\eta) = \left. \frac{\partial \phi}{\partial z} \right|_{z=\eta}, \;\; z = \eta \tag{12}$$

Boundary conditions

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Approximations:

- $\boldsymbol{u} \cdot \nabla \eta$ is 'small' compared to other terms,
- $\left. \frac{\partial \phi}{\partial z} \right|_{z=\eta} = \left. \frac{\partial \phi}{\partial z} \right|_{z=0} + \eta \left. \frac{\partial^2 \phi}{\partial z^2} \right|_{z=0} + \cdots \approx \left. \frac{\partial \phi}{\partial z} \right|_{z=0}$

Thus, (12) can be written as

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}, \;\; z = 0 \tag{13}$$

Boundary conditions

$$p = 0, \;\; z = \eta \tag{14}$$

Considering inviscid Navier-Stokes (Euler equation) in z-direction,

Boundary conditions

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Considering inviscid Navier-Stokes (Euler equation) in z-direction,

$$\begin{aligned} \rho \frac{\partial u_z}{\partial t} &= -\frac{\partial p}{\partial z} - \rho g \\ \Rightarrow \rho \frac{\partial^2 \phi}{\partial t \partial z} &= -\frac{\partial p}{\partial z} - \rho g \\ \Rightarrow \frac{\partial \phi}{\partial t} &= -\frac{p}{\rho} - gz \end{aligned} \tag{15}$$

Boundary conditions

At $z = \eta$, $p = 0$, so

$$\frac{\partial \phi}{\partial t} = -g\eta \tag{16}$$

$\dfrac{\partial \phi}{\partial t}$ at $z = \eta$ can be approximated by its value at $z = 0$,

$$\frac{\partial \phi}{\partial t} = -g\eta, \;\; z = 0 \tag{17}$$

Problem recap

To solve:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{18}$$

with boundary conditions:

$$\frac{\partial \phi}{\partial z} = 0,\ z = -H \tag{19}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t},\ z = 0 \tag{20}$$

$$\frac{\partial \phi}{\partial t} = -g\eta,\ z = 0 \tag{21}$$

Solution

$$\phi = f(z)e^{\mathrm{i}(kx-\omega t)} \tag{22}$$

$\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0$ gives us

$$\begin{aligned} -k^2 f + \frac{d^2 f}{dz^2} &= 0 \\ f &= C_1 e^{kz} + C_2 e^{-kz} \end{aligned} \tag{23}$$

Solution

At, $z = -H$, $\dfrac{\partial \phi}{\partial z} = 0$,

$$C_1 e^{-kH} - C_2 e^{kH} = 0 \tag{24}$$

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Also, at $z = 0$, $\dfrac{\partial \phi}{\partial z} = \dfrac{\partial \eta}{\partial t}$. If a is the amplitude of η,

$$\eta = a e^{\mathrm{i}(kx - \omega t)} \tag{25}$$

which gives us

$$k(C_1 - C_2) = -\mathrm{i} a \omega \tag{26}$$

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Solving for C_1 and C_2 gives us:

$$C_1 = -\frac{\mathrm{i} a \omega}{k(1 - e^{-2kH})},\ C_2 = -\frac{\mathrm{i} a \omega e^{-2kH}}{k(1 - e^{-2kH})} \tag{27}$$

Solution

$$
\begin{aligned}
f &= -\frac{\mathrm{i}a\omega}{k}\frac{e^{kz}+e^{-2kH}e^{-kz}}{1-e^{-2kH}} \\
&= -\frac{\mathrm{i}a\omega}{k}\frac{e^{k(z+H)}+e^{-k(z+H)}}{e^{kH}-e^{-kH}} \\
&= -\frac{\mathrm{i}a\omega}{k}\frac{\cosh(k(z+H))}{\sinh(kH)}
\end{aligned}
\tag{28}
$$

Solution

Thus,

$$\phi = -\frac{\mathrm{i}a\omega}{k}\frac{\cosh(k(z+H))}{\sinh(kH)}e^{\mathrm{i}(kx-\omega t)} \tag{29}$$

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Taking real part,

$$\phi = \frac{a\omega}{k}\frac{\cosh(k(z+H))}{\sinh(kH)}\sin(kx-\omega t) \tag{30}$$

Solution

Thus,

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Taking real part,

$$\phi = \frac{a\omega}{k}\frac{\cosh(k(z+H))}{\sinh(kH)}\sin(kx-\omega t) \tag{30}$$

which gives,

$$\begin{aligned} u_x &= \frac{\partial \phi}{\partial x} = a\omega\frac{\cosh(k(z+H))}{\sinh(kH)}\cos(kx-\omega t) \\ u_z &= \frac{\partial \phi}{\partial z} = a\omega\frac{\sinh(k(z+H))}{\sinh(kH)}\sin(kx-\omega t) \end{aligned} \tag{31}$$

Dispersion relation

Using $\frac{\partial \phi}{\partial t} = -g\eta$ at $z = 0$

$$\omega = \sqrt{gk \tanh(kH)} \tag{32}$$

Particle trajectories

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= u_x = a\omega \frac{\cosh(k(z_0 + H))}{\sinh(kH)} \cos(kx_0 - \omega t) \\
\frac{\partial z}{\partial t} &= u_z = a\omega \frac{\sinh(k(z_0 + H))}{\sinh(kH)} \sin(kx_0 - \omega t)
\end{aligned}
\tag{33}
$$

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Integrating,

$$\begin{aligned}x &= -a\frac{\cosh(k(z_0+H))}{\sinh(kH)}\sin(kx_0-\omega t)\\ z &= a\frac{\sinh(k(z_0+H))}{\sinh(kH)}\cos(kx_0-\omega t)\end{aligned} \tag{34}$$

Particle trajectories

$$\frac{x^2}{\left[a\cosh(k(z+H))/\sinh(kH)\right]^2}+\frac{z^2}{\left[a\sinh(k(z+H))/\sinh(kH)\right]^2}=1 \tag{35}$$

Particle trajectories

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which is the equation for an ellipse: $\dfrac{x^2}{A^2}+\dfrac{z^2}{B^2}=1.$

Particle trajectories

- Semi-major axis $A = a\dfrac{\cosh(k(z_0 + H))}{\sinh(kH)}$
- Semi-minor axis $B = a\dfrac{\sinh(k(z_0 + H))}{\sinh(kH)}$
- Distance between foci $= \sqrt{A^2 - B^2} = \dfrac{a}{\sinh(kH)} =$ constant
- Phase $= kx_0 - \omega t =$ same for all particles at same x_0

Particle trajectories

Movie ...

Streamlines

$$\frac{\partial \psi}{\partial z} = u_x \; ; \; \frac{\partial \psi}{\partial x} = -u_z \tag{36}$$

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$$\psi = \frac{a\omega}{k} \frac{\sinh(k(z+H))}{\sinh(kH)} \cos(kx - \omega t) \tag{37}$$

Streamlines

Movie ...

Phase speed

$$\omega = \sqrt{gk\tanh(kH)} \tag{38}$$

$$c = \frac{\omega}{k} = \frac{\sqrt{gk\tanh kH}}{k} = \sqrt{\frac{g}{k}\tanh kH} \tag{39}$$

Deep water approximation

$$kH >> 1, \;\; H > 0.28\lambda \tag{40}$$

$$\begin{aligned} A &= a\frac{\cosh(k(z_0+H))}{\sinh(kH)} \approx e^{kz_0} \\ B &= a\frac{\sinh(k(z_0+H))}{\sinh(kH)} \approx e^{kz_0} \\ \tanh kH &\approx 1 \end{aligned} \tag{41}$$

$A = B$, particles go in circles.

$$\begin{aligned} \omega &= \sqrt{gk} \\ c &= \frac{\omega}{k} = \sqrt{\frac{g}{k}} \end{aligned} \tag{42}$$

Shallow water approximation

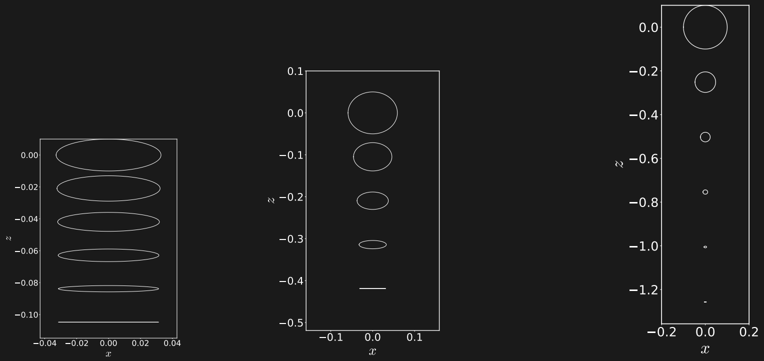
$$kH << 1,\ \ H < 0.07\lambda \tag{43}$$

$$\begin{aligned} A &= a\frac{\cosh(k(z_0+H))}{\sinh(kH)} \approx \frac{a}{kH} \\ B &= a\frac{\sinh(k(z_0+H))}{\sinh(kH)} \approx a\left(1+\frac{z}{H}\right) \\ \tanh kH &\approx kH \end{aligned} \tag{44}$$

$A =$constant, particles go in ellipses, with same semi-major axis. Semi-minor axes gets smaller.

$$\begin{aligned} \omega &= \sqrt{gk^2H} \\ c &= \frac{\omega}{k} = \sqrt{gH} \end{aligned} \tag{45}$$

Illustration



Two interacting waves

$$\eta = a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t) \tag{46}$$

Trig 101

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B \end{aligned} \tag{47}$$

Adding the two

$$2\cos A \cos B = \cos(A+B) + \cos(A-B) \tag{48}$$

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Replace $A \to \dfrac{a+b}{2},\ B \to \dfrac{a-b}{2}$

$$2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) = \cos a + \cos b \tag{49}$$

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$$\eta = 2a\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos\left(kx - \omega t\right) \tag{51}$$

where,

$$\begin{aligned}
\Delta k &= k_1 - k_2 \\
\Delta\omega &= \omega_1 - \omega_2 \\
k &= \frac{1}{2}(k_1 + k_2) \\
\omega &= \frac{1}{2}(\omega_1 + \omega_2)
\end{aligned} \tag{52}$$

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$$\eta = 2a \cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right) \cos\left(kx - \omega t\right) \tag{53}$$

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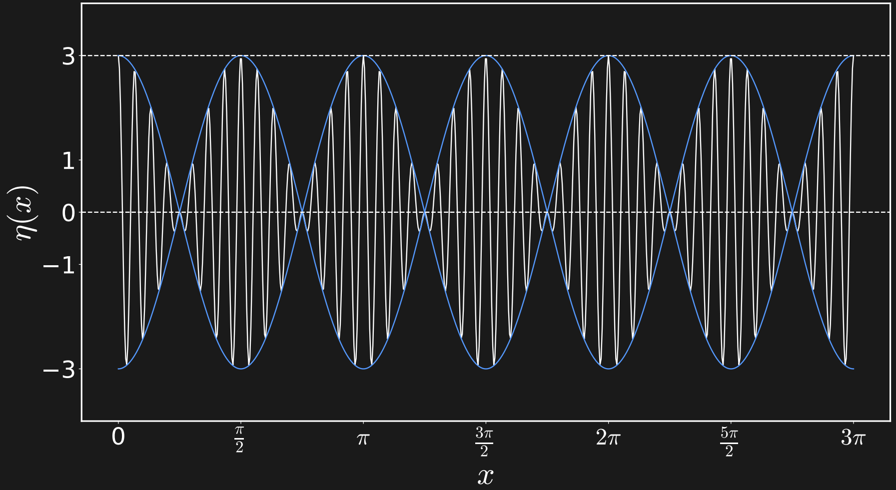
Group velocity



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Velocity of amplitude modulation: $c_g = \dfrac{\Delta\omega}{\Delta k}$

For a continuous distribution, $\omega(k)$,

$$c_g = \frac{d\omega}{dk} \tag{56}$$

Surface gravity waves

$$\omega(k) = \sqrt{gk \tanh(kH)} \tag{57}$$

$$c_g = \frac{d\omega}{dk} = \frac{c}{2}\left(1 + \frac{2kH}{\sinh(2kH)}\right) \tag{58}$$