Gauss coefficients and potential extrapolation

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1 Obtaining poloidal potential from Gauss coefficients

In an insulating region, current $\boldsymbol{J} = \nabla \times \boldsymbol{B} = 0$. Thus, if $\boldsymbol{B} = \nabla \times \nabla \times \mathcal{P}\hat{\boldsymbol{r}}$,

$$B_r = \sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell} \frac{\ell(\ell+1)}{r^2}\mathcal{P}_{\ell m} Y_\ell^m(\theta,\phi) \ . \tag{1}$$

In addition, writing $\boldsymbol{B} = -\nabla V$,

$$V = a\sum_{\ell=1}^{\infty}\sum_{m=0}^{\ell} \left(\frac{a}{r}\right)^{\ell+1} [g_{\ell m}\cos(m\phi) + h_{\ell m}\sin(m\phi)] P_\ell^m(\cos\theta) \ , \tag{2}$$

where a is the reference radius where the Gauss coefficients are given (usually surface of the planet). This implies:

$$\begin{aligned}
B_r = -\frac{\partial V}{\partial r} &= \sum (\ell+1)\left(\frac{a}{r}\right)^{\ell+2} N_{\ell m}[g_{\ell m}\cos(m\phi) + h_{\ell m}\sin(m\phi)] P_\ell^m(\cos\theta) \\
&= \sum (\ell+1)\left(\frac{a}{r}\right)^{\ell+2} N_{\ell m}[(g_{\ell m} - i h_{\ell m}) e^{im\phi}] P_\ell^m(\cos\theta) \\
&= \sum (\ell+1)\left(\frac{a}{r}\right)^{\ell+2} N_{\ell m}[(g_{\ell m} - i h_{\ell m})] Y_\ell^m(\cos\theta)
\end{aligned} \tag{3}$$

where, $N_{\ell m}$ is due to the Schmidt semi-normalization for Gauss coefficients,

$$N_{\ell m} = \begin{cases} \left(\dfrac{4\pi}{2\ell+1}\right)^{1/2}, & m = 0 \\ \left(\dfrac{2\pi}{2\ell+1}\right)^{1/2}, & m \neq 0 \ . \end{cases} \tag{4}$$

Comparing (1) and (3), we get,

$$\begin{aligned}
\frac{\ell(\ell+1)}{r^2}\mathcal{P}_{\ell m} &= (\ell+1)\left(\frac{a}{r}\right)^{\ell+2} N_{\ell m}[(g_{\ell m} - i h_{\ell m})] \\
\Rightarrow \mathcal{P}_{\ell m} &= \frac{r^2}{\ell}\left(\frac{a}{r}\right)^{\ell+2} N_{\ell m}[(g_{\ell m} - i h_{\ell m})] \\
\Rightarrow \mathcal{P}_{\ell m} &= \frac{a^2}{\ell}\left(\frac{a}{r}\right)^{\ell} N_{\ell m}[(g_{\ell m} - i h_{\ell m})] \ .
\end{aligned} \tag{5}$$

Thus, at $r = a$,

$$\mathcal{P}_{\ell m}(r = a) = \tilde{\mathcal{P}}_{\ell m} = \frac{a^2}{\ell} N_{\ell m}[(g_{\ell m} - ih_{\ell m})] \ , \tag{6}$$

which gives us,

$$\mathcal{P}_{\ell m}(r) = \tilde{\mathcal{P}}_{\ell m} \left(\frac{a}{r}\right)^{\ell} \ . \tag{7}$$

This gives us the definitions:

$$\begin{aligned} Q_{\ell m} &= \frac{\ell(\ell+1)}{r^2} \mathcal{P}_{\ell m} \ , \\ S_{\ell m} &= \frac{1}{r} \frac{\partial \mathcal{P}_{\ell m}}{\partial r} = -\frac{\ell}{r^2} \tilde{\mathcal{P}}_{\ell m} \left(\frac{a}{r}\right)^{\ell} = -\frac{\ell}{r^2} \mathcal{P}_{\ell m} \ . \end{aligned} \tag{8}$$

1.1 Using r in the expansion

In the case when, $\boldsymbol{B} = \nabla \times \nabla \times \mathcal{P} \boldsymbol{r}$,

$$B_r = \sum \frac{\ell(\ell+1)}{r} \mathcal{P}_{\ell m} Y_\ell^m(\theta, \phi) \ . \tag{9}$$

We can rewrite (5) as,

$$\begin{aligned} \frac{\ell(\ell+1)}{r} \mathcal{P}_{\ell m} &= (\ell+1) \left(\frac{a}{r}\right)^{\ell+2} N_{\ell m}[(g_{\ell m} - ih_{\ell m})] \\ \Rightarrow \mathcal{P}_{\ell m} &= \frac{r}{\ell} \left(\frac{a}{r}\right)^{\ell+2} N_{\ell m}[(g_{\ell m} - ih_{\ell m})] \\ \Rightarrow \mathcal{P}_{\ell m} &= \frac{a}{\ell} \left(\frac{a}{r}\right)^{\ell+1} N_{\ell m}[(g_{\ell m} - ih_{\ell m})] \ . \end{aligned} \tag{10}$$

Thus, at $r = a$,

$$\mathcal{P}_{\ell m}(r = a) = \tilde{\mathcal{P}}_{\ell m} = \frac{a}{\ell} N_{\ell m}[(g_{\ell m} - ih_{\ell m})] \ , \tag{11}$$

which gives us,

$$\mathcal{P}_{\ell m}(r) = \tilde{\mathcal{P}}_{\ell m} \left(\frac{a}{r}\right)^{\ell+1} \ . \tag{12}$$

This gives us the definitions:

$$\begin{aligned} Q_{\ell m} &= \frac{\ell(\ell+1)}{r} \mathcal{P}_{\ell m} \ , \\ S_{\ell m} &= \frac{1}{r} \frac{\partial}{\partial r} r \mathcal{P}_{\ell m} = -a \frac{\ell+1}{r^2} \tilde{\mathcal{P}}_{\ell m} \left(\frac{a}{r}\right)^{\ell} = -\frac{\ell+1}{r} \mathcal{P}_{\ell m} \ . \end{aligned} \tag{13}$$

2 Obtaining Gauss coefficients from poloidal potential

This time, we work with (1),

$$
\begin{aligned}
B_r &= \sum \frac{\ell(\ell+1)}{r^2}\mathcal{P}_{\ell m} Y_\ell^m(\theta,\phi) \\
&= \sum \frac{\ell(\ell+1)}{r^2}[\mathrm{Re}(\mathcal{P}_{\ell m}) + i\mathrm{Im}(\mathcal{P}_{\ell m})] P_\ell^m(\cos\theta) e^{im\phi} \\
&= \sum \frac{\ell(\ell+1)}{r^2}[\mathrm{Re}(\mathcal{P}_{\ell m})\cos(m\phi) - \mathrm{Im}(\mathcal{P}_{\ell m})\sin(m\phi)] P_\ell^m(\cos\theta)\ .
\end{aligned} \tag{14}
$$

Comparing against (3), we get,

$$
\frac{\ell(\ell+1)}{r^2}\mathrm{Re}(\mathcal{P}_{\ell m}) = (\ell+1)\left(\frac{a}{r}\right)^{\ell+2} N_{\ell m} g_{\ell m} \tag{15}
$$

$$
-\frac{\ell(\ell+1)}{r^2}\mathrm{Im}(\mathcal{P}_{\ell m}) = (\ell+1)\left(\frac{a}{r}\right)^{\ell+2} N_{\ell m} h_{\ell m}\ , \tag{16}
$$

which gives us,

$$
g_{\ell m} = \frac{1}{N_{\ell m}}\frac{\ell}{r^2}\left(\frac{r}{a}\right)^{\ell+2}\mathrm{Re}(\mathcal{P}_{\ell m}) = \frac{1}{N_{\ell m}}\frac{\ell}{a^2}\left(\frac{r}{a}\right)^{\ell}\mathrm{Re}(\mathcal{P}_{\ell m}) \tag{17}
$$

$$
h_{\ell m} = -\frac{1}{N_{\ell m}}\frac{\ell}{r^2}\left(\frac{r}{a}\right)^{\ell+2}\mathrm{Im}(\mathcal{P}_{\ell m}) = -\frac{1}{N_{\ell m}}\frac{\ell}{a^2}\left(\frac{r}{a}\right)^{\ell}\mathrm{Im}(\mathcal{P}_{\ell m}) \tag{18}
$$

2.1 Using $\boldsymbol{r}$ in the expansion

When, $\boldsymbol{B} = \nabla\times\nabla\times\mathcal{P}\boldsymbol{r}$,

$$
B_r = \sum \frac{\ell(\ell+1)}{r}[\mathrm{Re}(\mathcal{P}_{\ell m})\cos(m\phi) - \mathrm{Im}(\mathcal{P}_{\ell m})\sin(m\phi)] P_\ell^m(\cos\theta)\ , \tag{19}
$$

which yields,

$$
g_{\ell m} = \frac{1}{N_{\ell m}}\frac{\ell}{r}\left(\frac{r}{a}\right)^{\ell+2}\mathrm{Re}(\mathcal{P}_{\ell m}) = \frac{1}{N_{\ell m}}\frac{\ell}{a}\left(\frac{r}{a}\right)^{\ell+1}\mathrm{Re}(\mathcal{P}_{\ell m}) \tag{20}
$$

$$
h_{\ell m} = -\frac{1}{N_{\ell m}}\frac{\ell}{r}\left(\frac{r}{a}\right)^{\ell+2}\mathrm{Im}(\mathcal{P}_{\ell m}) = -\frac{1}{N_{\ell m}}\frac{\ell}{a}\left(\frac{r}{a}\right)^{\ell+1}\mathrm{Im}(\mathcal{P}_{\ell m}) \tag{21}
$$