Conservation Laws III : Energy conservation

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1 Total energy equation

Conservation of energy requires that the change in total energy dE of a collection of fluid parcels comes from change in heat energy dQ and from the contribution of work done dW by all the forces,

$$dE = dQ + dW\,, \tag{1}$$

which gives,

$$\frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}\,. \tag{2}$$

Simplifying the LHS

The total energy for a fixed collection of fluid particles is the sum of the kinetic energy $\left(\frac{1}{2}\rho u^2\right)$ and the internal energy ρe integrated over the volume, i.e.,

$$\frac{dE}{dt} = \int_V \frac{d}{dt}\rho\left(\frac{1}{2}u^2 + e\right)dV = \frac{d}{dt}\int_V \rho e_0\,, \tag{3}$$

where, $e_0 = \frac{1}{2}u^2 + e$. Using Reynolds transport theorem we get,

$$\begin{aligned}
\frac{dE}{dt} &= \int_V \frac{\partial}{\partial t}(\rho e_0)dV + \int_S \rho e_0 \boldsymbol{u}\cdot\hat{\boldsymbol{n}}dS \\
&= \int_V \frac{\partial}{\partial t}(\rho e_0)dV + \int_V \nabla\cdot(\rho\boldsymbol{u}e_0)dV \\
&= \int_V \left[e_0\frac{\partial\rho}{\partial t} + \rho\frac{\partial e_0}{\partial t} + e_0\nabla\cdot(\rho\boldsymbol{u}) + \rho\boldsymbol{u}\cdot\nabla e_0\right]dV \\
&= \int_V \left[e_0\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right) + \rho\left(\frac{\partial e_0}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)\right]dV\,,
\end{aligned} \tag{4}$$

which gives

$$\frac{dE}{dt} = \int_V \rho\left(\frac{\partial e_0}{\partial t} + \boldsymbol{u}\cdot\nabla e_0\right)dV = \int_V \rho\frac{De_0}{dt}dV\,, \tag{5}$$

where we have used the divergence theorem in the second step and mass conservation $\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right) = 0$ in the last step.

Deriving an expression for dQ/dt

The change in thermal energy in the system can be quantified by noticing that any change takes place due to heat entering or leaving the collection of fluid parcels through the surface of the volume. Thus,

$$\frac{dQ}{dt} = -\int_S \boldsymbol{q} \cdot \hat{\boldsymbol{n}} dS = -\int_V \nabla \cdot \boldsymbol{q} dV \,, \tag{6}$$

where $\boldsymbol{q}$ is the heat flux through the surface and the negative sign signifies the fact that the normal to the surface points outwards while increase in thermal energy happens due to heat *entering* the system in the opposite direction, inwards. The second step is derived using the divergence theorem. Using Fourier's Law of heat conduction, we can write,

$$\boldsymbol{q} = -k\nabla T \,, \tag{7}$$

where, ∇T is the local gradient of temperature and k is the thermal conductivity dependent on the material. This basically says that heat travels down the temperature gradient, from hot to cold. Substituting this, we obtain,

$$\frac{dQ}{dt} = \int_V \nabla \cdot (k\nabla T) dV \tag{8}$$

Deriving an expression for dW/dt

Work w done by a force $\boldsymbol{F}$ is given by

$$w = \boldsymbol{F} \cdot \boldsymbol{r} \tag{9}$$

where, $\boldsymbol{r}$ is the displacement due to the force. Thus, the rate of work done is,

$$\frac{dw}{dt} = \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = \boldsymbol{F} \cdot \boldsymbol{u} \,. \tag{10}$$

Recall while deriving the equation of motion, we had two categories of forces, body forces, $\boldsymbol{F}_b$ and surface stresses, $\boldsymbol{\tau}$. Thus, the total work done by these forces is given by,

$$\frac{dW}{dt} = \int_V \boldsymbol{F}_b \cdot \boldsymbol{u} dV + \int_S \boldsymbol{u} \cdot \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} dS = \int_V \left(\boldsymbol{F}_b \cdot \boldsymbol{u} + \nabla \cdot (\boldsymbol{u} \cdot \boldsymbol{\tau})\right) dV \,. \tag{11}$$

We can decompose the second term into contributions from pressure p and deviatoric stresses $\boldsymbol{\tau}$ (note the difference, $\boldsymbol{\tau}$ vs $\boldsymbol{\tau}$),

$$\nabla \cdot (\boldsymbol{u} \cdot \boldsymbol{\tau}) = \nabla \cdot (\boldsymbol{u} \cdot (-p + \boldsymbol{\tau})) = -p(\nabla \cdot \boldsymbol{u}) - \boldsymbol{u} \cdot \nabla p + \nabla \cdot (\boldsymbol{u} \cdot \boldsymbol{\tau}) \,, \tag{12}$$

where we have used the chain rule for the product of a scalar and a vector, $\nabla \cdot (f\boldsymbol{a}) = f\nabla \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \nabla f$. The last term can be decomposed into two parts,

$$\nabla \cdot (\boldsymbol{u} \cdot \boldsymbol{\tau}) = \frac{\partial}{\partial x_j}(u_i \tau_{ij}) = \tau_{ij}\frac{\partial u_j}{\partial x_j} + u_j \frac{\partial \tau_{ij}}{\partial x_j} \,. \tag{13}$$

The first part of this represents deformation of the volume due to surface stresses that contributes to the *internal energy* while the second part represents work done by the surface stresses to increase the *kinetic energy* of the fluid parcel. Recall that $\frac{\partial u_j}{\partial x_j} = S_{ij} + R_{ij}$, where $S_{ij} = e_{ij} =$

$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$ and $R = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)$ are the symmetric and antisymmetric parts of the velocity gradient tensor, representing rate of strain and rotation, respectively. The dot product of a symmetric tensor (τ_{ij}) and an antisymmetric tensor R_{ij} is zero. Hence,

$$\nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{\tau}) = \frac{\partial}{\partial x_j}(u_i\tau_{ij}) = \tau_{ij}S_{ij} + +u_j\frac{\partial \tau_{ij}}{\partial x_j} = \tau_{ij}e_{ij} + u_j\frac{\partial \tau_{ij}}{\partial x_j} = \boldsymbol{\tau}:\mathbf{S} + \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau})\,. \tag{14}$$

Using the constitutive equation for the deviatoric stress $\boldsymbol{\tau}$ for a Newtonian fluid,

$$\tau_{ij} = 2\mu e_{ij} - \frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})\delta_{ij}\,, \tag{15}$$

we obtain,

$$\Phi_\mu = \tau_{ij}e_{ij} = (2\mu e_{ij} - \frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})\delta_{ij})e_{ij} = 2\mu e_{ij}e_{ij} - \frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})^2 = 2\mu\left(e_{ij} - \frac{1}{3}\nabla\cdot\boldsymbol{u}\right)^2\,. \tag{16}$$

The above equation provides an expression for the rate of deformation work done by viscous stresses. Thus, we obtain the expression for dW/dt as,

$$\frac{dW}{dt} = \int_V \Bigg[\underbrace{\boldsymbol{F}_b\cdot\boldsymbol{u}}_{\text{Work done by body forces}} + \underbrace{\boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau})}_{\text{Increase in KE by surface stresses}} - \underbrace{p(\nabla\cdot\boldsymbol{u})}_{\text{Work done due to contraction or expansion}} - \underbrace{\boldsymbol{u}\cdot\nabla p}_{\text{Work done by pressure gradient}} + \underbrace{\Phi_\mu}_{\text{Deformation work by viscous stresses}}\Bigg] dV \tag{17}$$

Bringing it together

Using the expressions for $dE/dt, dQ/dt$ and dW/dt, we obtain,

$$\frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}$$
$$\Rightarrow \int_V \rho\frac{D}{Dt}\left(\frac{1}{2}\rho u^2 + \rho e\right) dV = \int_V \left[\nabla\cdot(k\nabla T) + \boldsymbol{F}_b\cdot\boldsymbol{u} + \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) - p(\nabla\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\nabla p + \Phi_\mu\right] dV \tag{18}$$

This is the energy equation for total energy of a collection of fluid parcels. Since it's valid for any arbitrary volume at all times, we can remove the integral signs and equate the integrands. We obtain,

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\rho u^2 + \rho e\right) = \nabla\cdot(k\nabla T) + \boldsymbol{F}_b\cdot\boldsymbol{u} + \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) - p(\nabla\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\nabla p + \Phi_\mu \tag{19}$$

1.1 Mechanical energy equation

The equation for mechanical energy can be derived separately by performing a dot product of velocity on both sides of the equation of motion,

$$\boldsymbol{u} \cdot \rho \frac{D\boldsymbol{u}}{Dt} = \boldsymbol{u} \cdot (-\nabla p + \boldsymbol{F}_b + \nabla \cdot \boldsymbol{\tau}) \,, \tag{20}$$

which gives,

$$\frac{D}{Dt}\left(\frac{1}{2}\rho u^2\right) = -\boldsymbol{u} \cdot \nabla p + \boldsymbol{F}_b \cdot \boldsymbol{u} + \boldsymbol{u} \cdot (\nabla \cdot \boldsymbol{\tau}) \,. \tag{21}$$

1.2 Internal energy equation

Subtracting the mechanical energy equation (21) from the total energy equation (19), we obtain the internal energy equation,

$$\rho \frac{De}{Dt} = \nabla \cdot (k \nabla T) - p(\nabla \cdot \boldsymbol{u}) + \Phi_\mu \tag{22}$$

An interesting note

We can rewrite the mechanical energy equation (21), as

$$\begin{aligned}\frac{D}{Dt}\left(\frac{1}{2}\rho u^2\right) &= \boldsymbol{F}_b \cdot \boldsymbol{u} + \boldsymbol{u} \cdot (-\nabla p + \nabla \cdot \boldsymbol{\tau}) \\ &= \boldsymbol{F}_b \cdot \boldsymbol{u} + \nabla \cdot (-\boldsymbol{u}p + \boldsymbol{u} \cdot \boldsymbol{\tau}) - (-p(\nabla \cdot \boldsymbol{u}) + \boldsymbol{\tau} : \mathsf{S}) \\ &= \boldsymbol{F}_b \cdot \boldsymbol{u} + \nabla \cdot (-\boldsymbol{u}p + \boldsymbol{u} \cdot \boldsymbol{\tau}) + p(\nabla \cdot \boldsymbol{u}) - \Phi_\mu \,.\end{aligned} \tag{23}$$

The last two terms also appear in the internal energy equation but with opposite signs. The term $p(\nabla \cdot \boldsymbol{u})$ represents reversible work done by pressure or on pressure when the volume of collection of fluid particles contracts or expands, this leads to an increase or decrease in internal energy, respectively. The viscous dissipation term Φ_μ represents heat lost due to viscous friction and it gives rise to heat - thus also converting mechanical energy to internal or thermal energy.

1.3 Change of variable to entropy, s

Here, we focus on a change of variable to entropy, s. We start with the first law of thermodynamics,

$$de = Tds - pdv = Tds - pd\frac{1}{\rho} = Tds + \frac{p}{\rho^2}d\rho \,. \tag{24}$$

Substituting this in the internal energy equation (22), we obtain,

$$\rho T \frac{Ds}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt} = \nabla \cdot (k \nabla T) - p(\nabla \cdot \boldsymbol{u}) + \Phi_\mu \,. \tag{25}$$

We notice that,

$$
\begin{aligned}
\frac{p}{\rho}\frac{D\rho}{Dt} &= \frac{p}{\rho}\left[\frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho\right] \\
&= \frac{p}{\rho}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) - \rho(\nabla\cdot\boldsymbol{u})\right] \\
&= -p(\nabla\cdot\boldsymbol{u})\,.
\end{aligned} \tag{26}
$$

Substituting in (25), we obtain,

$$
\rho T\frac{Ds}{Dt} = \nabla\cdot(k\nabla T) + \Phi_\mu\,. \tag{27}
$$

The thermal energy equation is often stated this way. We can make a further change of variables in terms of temperature T and pressure p using $ds = \left(\frac{\partial s}{\partial T}\right)_p dT + \left(\frac{\partial s}{\partial p}\right)_T dp = \frac{C_p}{T}dT - \frac{\alpha_T}{\rho}dp$, where C_p is specific heat at constant pressure and α_T is the thermal expansion coefficiant. We get,

$$
\rho C_p\frac{DT}{Dt} - \alpha_T T\frac{Dp}{Dt} = \nabla\cdot(k\nabla T) + \Phi_\mu\,. \tag{28}
$$