Conservation Laws II : The rotating frame

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Recall that the equations of motion were derived with respect to an *inertial* or non-accelerating frame. In such a frame, objects with no forces acting on them stay at rest or move with a uniform velocity in a straight line. In a non-inertial or a frame accelerating with respect to an inertial frame, this is not the case. Observations of the Earth's atmosphere and oceans are usually made with respect to the Earth's surface, which is rotating. Thus, these observations take place in a rotating frame of reference, which is non-inertial. Thus, we have to find suitable modifications to our equations of motion for a rotating frame.

1 Non-inertial frames

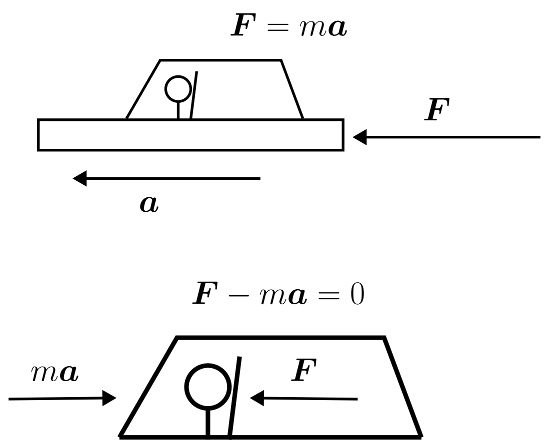

Figure 1: Acceleration of a person in a car as seen from an inertial (top) frame and a non-inertial (bottom) frame.

Think of a person A sitting in an accelerating car. From the frame of reference of a person B outside, the car seat is pushing A forward and A accelerates. From the point of view of A, they are at rest. But they feel a force pushing them back. This is due to the fact that A wants to remain at rest due to inertia while the car seat is pushing them forward. To make sense of the equation of motion of A from inside the car, one thus needs to include a *fictitious* (also called *inertial*) force which is opposite in direction to the acceleration of A as seen by a person standing outside. This is shown in figure 1. Note that in this case the two equations of motion end up being identical. This is not always the case.

2 Rotating frame

A rotating frame is an example of a non-inertial frame too. Even if the angular speed of rotation is constant, the direction of the coordinate system is constantly changing, which makes it an

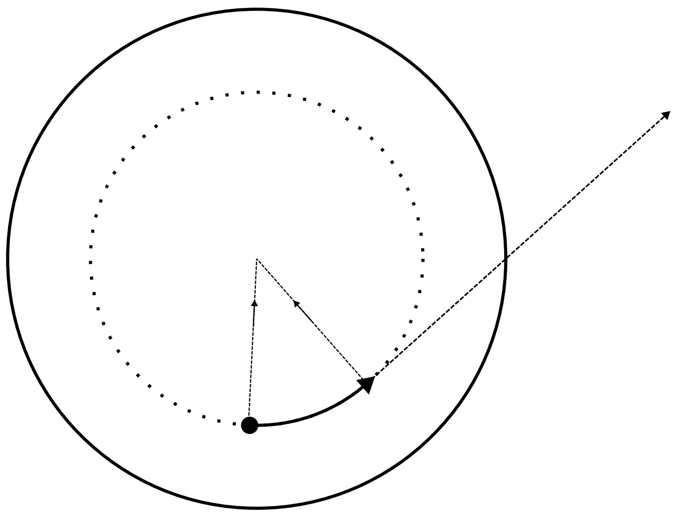

Figure 2: Schematic depiction of centrifugal force. This is a top view of a merry-go-round. The black dot is a person standing on it.

accelerating frame. The two most important effects observed in a rotating frame are:

Centrifugal force

Imagine being on a merry-go-round. When it rotates fast, we can feel ourselves getting pushed back. Figure 2 shows a top view. The person rotating along with the merry-go-round would actually follow the trajectory shown with the long dashed line in the absence of any external force. However, the person is usually holding on to something (hopefully) to prevent that from happening. This forces an acceleration directed inward at each moment, the net result of which is the person staying along the dotted circular path.

Coriolis effect

Imagine the same person A standing on the merry-go-round. The whole disk of the merry-go-round rotates as one solid object. This means that points farther away from the center have to cover a larger distance in the same time and thus have to move faster compared to points closer to the center. Imagine, a second person B (shown by orange dot) is standing closer to the center than A, going around on a smaller circle. B throws a ball (shown in blue) towards person A. However, because person A is moving along the larger circular path, they are going faster than the ball's speed in the tangential direction and are thus, unable to catch it. From an inertial frame, just looking from top, an observer would see A moving faster than the ball can catch up and thus, the ball lands to the left of A. However, from a rotating frame it would look like the ball went along a trajectory that curved to the *right* and landed to the left of A while both A and B stayed stationary. Thus, to write down the equation of motion of the ball in the rotating frame, A and B have to account for a force that is making the ball deflect to the *right* - a direction perpendicular to both the rotation axis (vertical) as well as the ball's velocity.

Note that the exact same thing will happen if A threw a ball towards B. In this case, B is going slower and the ball lands to their right. From a rotating frame, the ball again, got deflected to its *right* compared to its initial direction of motion. On a planet's surface, the exact same thing occurs. The entire Earth's surface rotates with a uniform angular velocity. This means, points

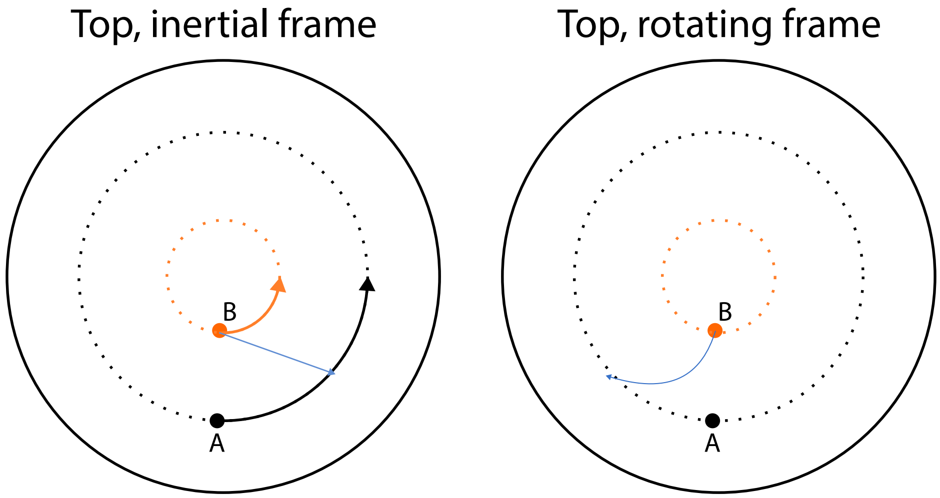

Figure 3: Schematic depiction of Coriolis effect. This is a top view of a merry-go-round. The black dot is person A while the orange dot is person B throwing a ball at person A, shown in blue. Left panel shows the view from an inertial frame, right panel shows the view from the point of view of A.

at the equator have a faster tangential velocity compared to those at the pole. Using the same logic as above to person B near the equator in figure 4 throwing a ball at person A away from the equator, we see that the ball lands to the right of A, seemingly getting deflected to its right. Note that this is always true in the northern hemisphere. In the southern hemisphere the opposite would happen - the ball would get deflected to its left.

2.1 Rate of change of a rotating vector

Consider a vector $\boldsymbol{A}$ with constant magnitude but rotating with angular velocity $\boldsymbol{\Omega}$. The angle between the two vectors is γ (Figure 5). In time Δt, $\boldsymbol{A}$ has rotated through an angle $\Delta\phi$. The total magnitude of change in the vector $\boldsymbol{A}$ is given by,

$$\boldsymbol{A}(t+\Delta t) - \boldsymbol{A}(t) = |\Delta \boldsymbol{A}| = (\text{projection of } \boldsymbol{A} \text{ onto the plane perpendicular to } \boldsymbol{\Omega})\Delta\phi\,, \tag{1}$$

which gives us,

$$|\Delta \boldsymbol{A}| = |\boldsymbol{A}| \sin\gamma \Delta\phi\,. \tag{2}$$

The assumption here is that $\Delta\phi$ is small enough that the arc length can be approximated by a straight line. This gets progressively accurate as one approaches infinitesimally small lengths. The direction of $\Delta\boldsymbol{A}$ is perpendicular to both $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ and can be written as,

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\Omega}\times\boldsymbol{A}}{|\boldsymbol{\Omega}\times\boldsymbol{A}|}\,, \tag{3}$$

which can be verified using the right hand rule for cross products. Thus, we can write the rate of change of $\boldsymbol{A}$ as,

$$\frac{d\boldsymbol{A}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{A}}{\Delta t} = \frac{1}{\Delta t}\hat{\boldsymbol{n}}|\boldsymbol{A}|\sin\gamma\Delta\phi = \frac{\boldsymbol{\Omega}\times\boldsymbol{A}}{|\boldsymbol{\Omega}\times\boldsymbol{A}|}|\boldsymbol{A}|\sin\gamma\frac{\Delta\phi}{\Delta t} = \boldsymbol{\Omega}\times\boldsymbol{A}\,, \tag{4}$$

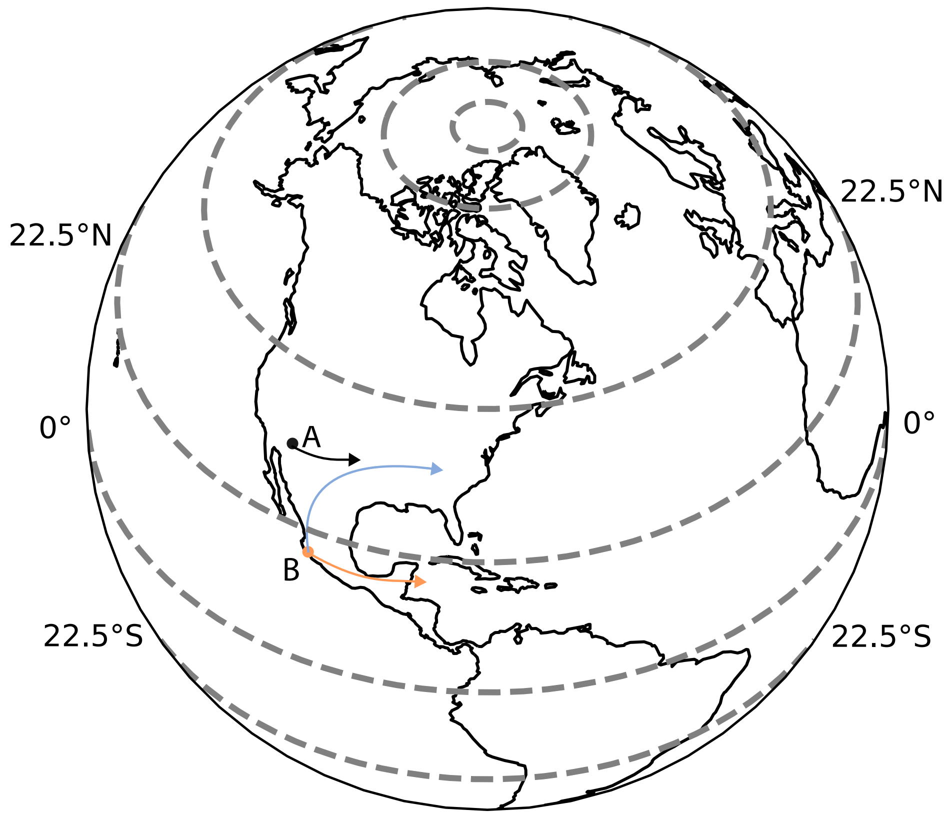

Figure 4: (Exaggerated) demonstration of Coriolis effect on Earth's surface.

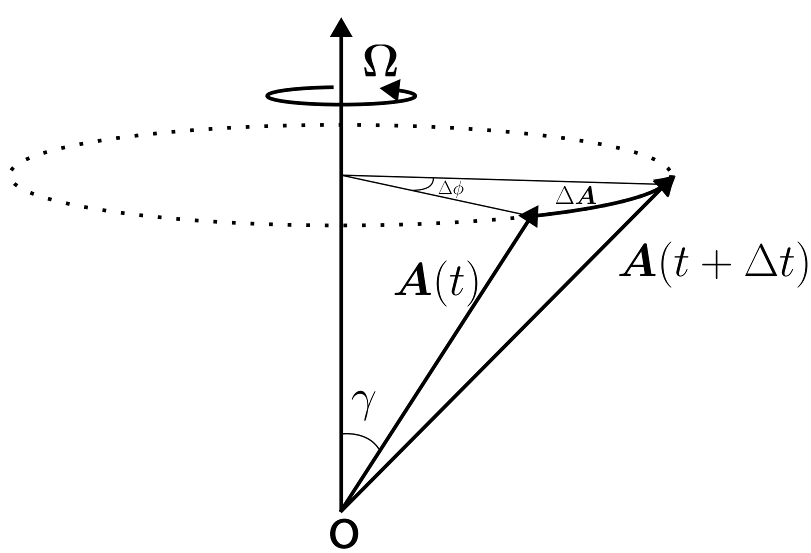

Figure 5: The change in a vector due to rotation.

where we have used $\dfrac{\Delta\phi}{\Delta t} = |\boldsymbol{\Omega}|$ and $|\boldsymbol{\Omega}\times\boldsymbol{A}| = |\boldsymbol{\Omega}||\boldsymbol{A}|\sin\gamma$. To conclude, the rate of change of a vector $\boldsymbol{A}$ of fixed magnitude, rotating with angular velocity $\boldsymbol{\Omega}$ is given by,

$$\frac{d\boldsymbol{A}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{A}\,. \tag{5}$$

2.2 Rate of change of a vector in a rotating frame

In the previous example, we looked at the acceleration due to the rotation of a vector. In this part, we will look at how the rate of change of a vector in a rotating coordinate system is related to that in an inertial frame. The difference is that we had held the coordinate system fixed in the previous example, here the coordinate frame is rotating with an angular velocity $\boldsymbol{\Omega}$. Consider a vector $\boldsymbol{B} = B_1\hat{\boldsymbol{x}}_1 + B_2\hat{\boldsymbol{x}}_2 + B_3\hat{\boldsymbol{x}}_3$ in this rotating coordinate system. The unit vectors in the three directions $(\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \hat{\boldsymbol{x}}_3)$ are thus time-dependent with respect to an inertial observer. In the rotating frame, the rate of change of $\boldsymbol{B}$ is given by,

$$\left(\frac{d\boldsymbol{B}}{dt}\right)_R = \frac{dB_1}{dt}\hat{\boldsymbol{x}}_1 + \frac{dB_2}{dt}\hat{\boldsymbol{x}}_2 + \frac{dB_3}{dt}\hat{\boldsymbol{x}}_3\,, \tag{6}$$

where subscript R denotes the rotating frame. The unit vectors do not change with time in the rotating frame. However, for a non-rotating, inertial observer in frame I, we would have,

$$\left(\frac{d\boldsymbol{B}}{dt}\right)_I = \left(\frac{dB_1}{dt}\hat{\boldsymbol{x}}_1 + \frac{dB_2}{dt}\hat{\boldsymbol{x}}_2 + \frac{dB_3}{dt}\hat{\boldsymbol{x}}_3\right) + B_1\frac{d\hat{\boldsymbol{x}}_1}{dt} + B_2\frac{d\hat{\boldsymbol{x}}_2}{dt} + B_3\frac{d\hat{\boldsymbol{x}}_3}{dt}\,. \tag{7}$$

The rate of change of the scalars B_i, $i = 1, 2, 3$, do not depend on the reference frame. Thus, the sum $\dfrac{dB_1}{dt}\hat{\boldsymbol{x}}_1 + \dfrac{dB_2}{dt}\hat{\boldsymbol{x}}_2 + \dfrac{dB_3}{dt}\hat{\boldsymbol{x}}_3$ is given by equation (6). For the other terms, we make use of equation (5) that we derived in the previous section and obtain,

$$\begin{aligned}\left(\frac{d\boldsymbol{B}}{dt}\right)_I &= \left(\frac{d\boldsymbol{B}}{dt}\right)_R + B_1(\boldsymbol{\Omega}\times\hat{\boldsymbol{x}}_1) + B_2(\boldsymbol{\Omega}\times\hat{\boldsymbol{x}}_2) + B_3(\boldsymbol{\Omega}\times\hat{\boldsymbol{x}}_3)\\ &= \left(\frac{d\boldsymbol{B}}{dt}\right)_R + \boldsymbol{\Omega}\times(B_1\hat{\boldsymbol{x}}_1 + B_2\hat{\boldsymbol{x}}_2 + B_3\hat{\boldsymbol{x}}_3)\\ &= \left(\frac{d\boldsymbol{B}}{dt}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{B}\,.\end{aligned} \tag{8}$$

Thus, the rate of change of a vector in the two frames, inertial and rotating, are related by

$$\left(\frac{d\boldsymbol{B}}{dt}\right)_I = \left(\frac{d\boldsymbol{B}}{dt}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{B}\,. \tag{9}$$

3 Equations of motion in a rotating frame

Consider the position vector $\boldsymbol{r}$ of a fluid particle. The rate of change of this position vector in an inertial and rotating frame are related by equation (9),

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_I = \left(\frac{d\boldsymbol{r}}{dt}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{r}\,. \tag{10}$$

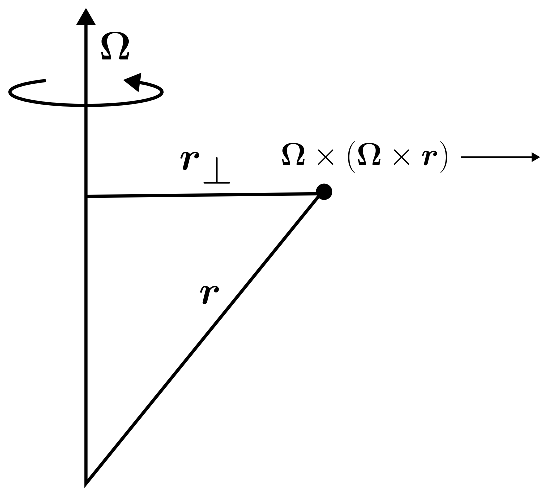

Figure 6: The cross product in the centrifugal acceleration can be written as $\boldsymbol{\Omega} \times \boldsymbol{r}_\perp$.

The time derivatives on the two sides are the velocities of the fluid particle in an inertial and rotating reference frame, respectively. Thus, we can write,

$$\boldsymbol{u}_I = \boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r}\,. \tag{11}$$

We derived the equations of motion for an inertial frame and need to relate the acceleration in an inertial frame to that in a rotating frame. Note that we cannot simply differentiate both sides of (11) to obtain this, since the frame of the derivative will not be well-defined. Thus, we make use of equation (9) once again for $\boldsymbol{u}_I$ and obtain,

$$\left(\frac{d\boldsymbol{u}_I}{dt}\right)_I = \left(\frac{d\boldsymbol{u}_I}{dt}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{u}_I\,. \tag{12}$$

Now making use of equation (11) and substituting $\boldsymbol{u}_I$, we obtain,

$$\begin{aligned}
\left(\frac{d\boldsymbol{u}_I}{dt}\right)_I &= \left(\frac{d}{dt}(\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r})\right)_R + \boldsymbol{\Omega} \times (\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \left(\frac{d\boldsymbol{u}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_R + \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r} + \boldsymbol{\Omega} \times \boldsymbol{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times r) \\
&= \left(\frac{d\boldsymbol{u}_R}{dt}\right)_R + 2\boldsymbol{\Omega} \times \boldsymbol{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r}\,,
\end{aligned} \tag{13}$$

Therefore,

$$\rho\left(\frac{d\boldsymbol{u}_R}{dt}\right)_R = \rho\left(\frac{d\boldsymbol{u}_I}{dt}\right)_I + \rho(-2\boldsymbol{\Omega} \times \boldsymbol{u}_R - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) - \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r})\,. \tag{14}$$

We get three additional forces on the right hand side to explain the difference in fluid accelerations in the two frames. Consequently, the *fictitious forces* that we need to add to our equation of motion in a rotating frame are,

- **Centrifugal force** $= -\rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$. This is the force that is experienced by a fluid parcel in a rotating frame that tends to move it away from the rotation axis. In an inertial frame,

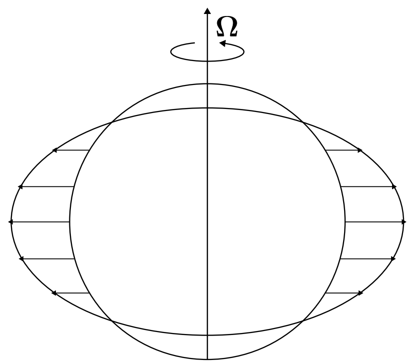

Figure 7: Deformation of a planet due to centrifugal force.

this is called the *centripetal* force which tries to constantly move a fluid parcel towards the rotation axis. From figure 6, we can see that the cross product $\boldsymbol{\Omega} \times \boldsymbol{r}$ can be written as $\boldsymbol{\Omega} \times \boldsymbol{r}_\perp$, since a cross product takes the product with the projection of a vector. Using this, we can write the centrifugal acceleration as,

$$-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_\perp) = (\boldsymbol{\Omega} \cdot \boldsymbol{r}_\perp)\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\boldsymbol{r}_\perp = |\boldsymbol{\Omega}|^2 \boldsymbol{r}_\perp \,, \tag{15}$$

where have made use of the relation of triple product of vectors,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \,. \tag{16}$$

The expression for the centrifugal force can be written as the gradient of a potential,

$$|\boldsymbol{\Omega}|^2 \boldsymbol{r}_\perp = \nabla \frac{1}{2}(|\boldsymbol{\Omega}|^2 |\boldsymbol{r}_\perp|^2) = \nabla \frac{1}{2}(|\boldsymbol{\Omega} \times \boldsymbol{r}_\perp|^2) = \nabla \Phi_{cent} \tag{17}$$

This gives us two ways to *absorb* this potential into the existing forces in our equation of motion:

1. **Modify pressure** by writing a new pressure gradient as $\nabla p' = \nabla p + \nabla(\rho \Phi_{cent})$ in case of incompressible fluids, or
2. **Modify gravity** by writing the new gravitational acceleration as $\boldsymbol{g}' = \boldsymbol{g} + |\boldsymbol{\Omega}|^2 \boldsymbol{r}_\perp$. Note that gravity can also be written in terms of a potential, $\boldsymbol{g} = -\nabla \Phi_g$ and the modification can also be done in terms of potentials, $\Phi'_g = \Phi_g - \Phi_{cent}$.

For a rotating planet, this implies a modification to the hydrostatic balance, making planets into 'oblate spheroids' than perfect spheres (figure 7). The centrifugal force, $|\boldsymbol{\Omega}|^2 \boldsymbol{r}_\perp$ acts to lower the effective gravity in a place, modifying the equipotential surfaces. Thus, gravity is stronger at the poles than the equator. The difference can be computed for Earth : $|\boldsymbol{\Omega}|^2 r_{eq} = \left(\dfrac{2\pi}{24 \times 3600}\right)^2 6400 \times 10^3\,\mathrm{m/s^2} = 3.4\,\mathrm{cm/s^2}$, which is about 0.5% of the standard value of $g = 9.81\,\mathrm{m/s^2}$.

- **Coriolis force** $= -2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$. The Coriolis force is the most significant modification to the equation of motion in a rotating frame. It deflects parcels of fluids in a direction perpendicular to both the fluid motion and the rotation axis. In the northern hemisphere of a planet, fluid parcels always get deflected to *the right*, while in the southern hemisphere, they always get

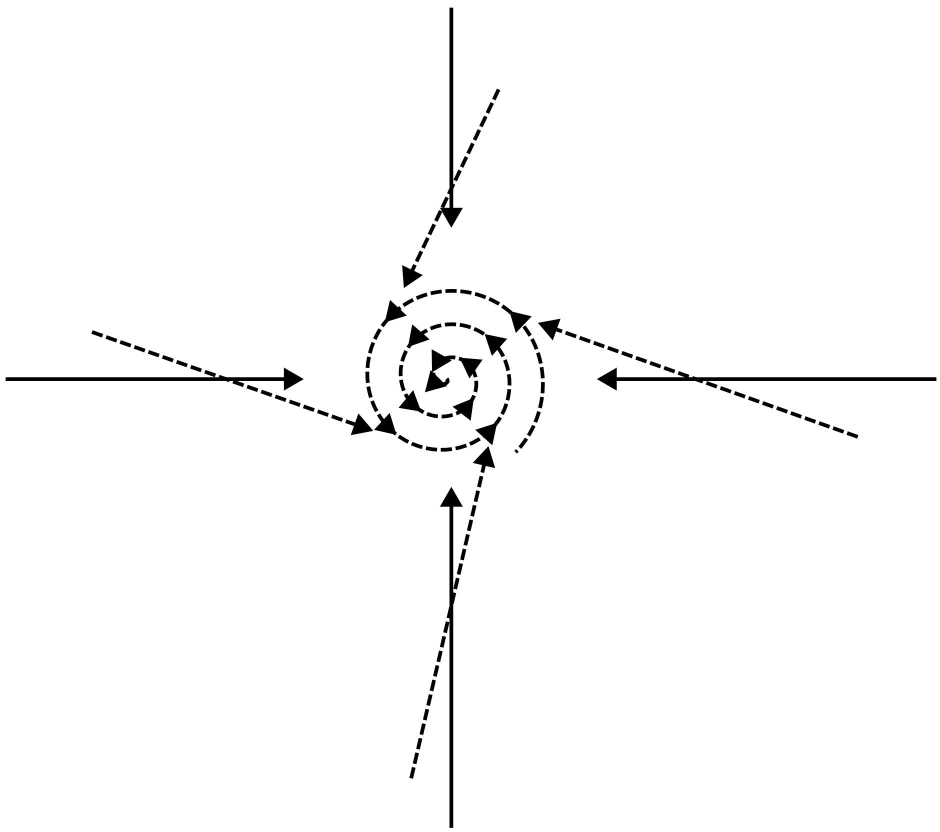

Figure 8: Formation of cyclone in the northern hemisphere.

deflected to *the left*. This causes cyclones in northern hemisphere to rotate counter-clockwise as shown in figure 8 while those in the southern hemisphere to rotate clockwise.

- **Poincaré force** $= -\rho \dfrac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r}$. This is cause when the rotation rate or axis vary with time. Usually used in the study of precessing bodies. For most applications, this is not needed.