Conservation Laws I

September 2023

1 Conservation of mass

1.1 Using a Cartesian control volume

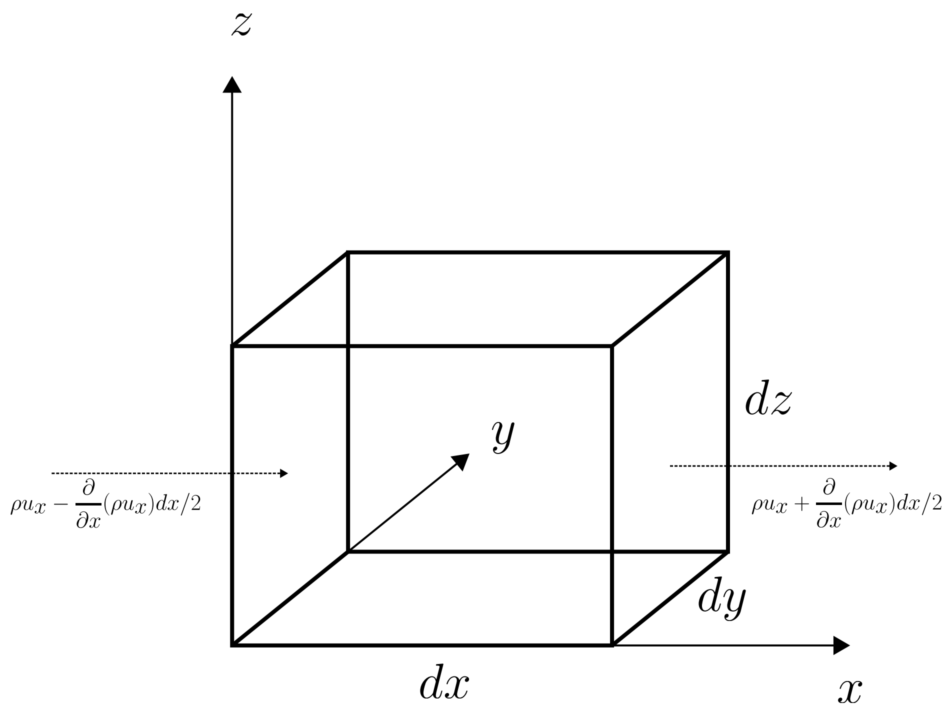

Figure 1: A cuboid control volume with axes along edges and lengths dx,dy and dz.

Consider a fluid with density ρ. Inside an infinitesimal cuboid 'control volume' V as shown in figure 1, the total mass of fluid is given by $\rho(dxdydz)$. If the flow velocity at the center of the volume in the x-direction is u_x, the rate at which mass flows into the volume is $\rho u_x(dydz)$ at the center, since $dydz$ is the surface area of the face along the y-z plane. On the right face away from the origin, the mass flow rate is given by that in the center and an additional contribution due to a gradient of mass flow rate along the x-direction:

$$\left[\rho u_x + \underbrace{\frac{\partial}{\partial x}(\rho u_x)dx/2}_{\text{Gradient times distance}}\right] dydz\,.$$

Similarly, on the left face near the origin, the mass flow rate is given by,

$$\left[\rho u_x - \underbrace{\frac{\partial}{\partial x}(\rho u_x)dx/2}_{\text{Gradient times distance}}\right] dydz\,.$$

Thus, we can write, for the x-direction:

$$\text{Total rate of mass flow, } x = \left[\rho u_x - \frac{\partial}{\partial x}(\rho u_x)dx/2\right] dydz - \left[\rho u_x + \frac{\partial}{\partial x}(\rho u_x)dx/2\right] dydz = -\frac{\partial}{\partial x}(\rho u_x)dxdydz\,.$$

Similarly, for y- and z-directions, we get

$$\text{Total rate of mass flow, } y = -\frac{\partial}{\partial y}(\rho u_y)dxdydz$$
$$\text{Total rate of mass flow, } z = -\frac{\partial}{\partial z}(\rho u_z)dxdydz$$

The statement of conservation of mass can be written simply as,

$$\text{Rate of change of mass} = \text{Total rate of mass flow}\,.$$

Recalling that total mass in the volume is given by $\rho(dxdydz)$, we can write the above as,

$$\frac{\partial}{\partial t}\rho(dxdydz) = -\left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right] dxdydz\,.$$

Since our control volume is fixed in time, we obtain,

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0\,. \tag{1}$$

In vector notation, this can be written as,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{2}$$

1.2 More formal derivation

Consider an arbitrary volume V fixed in space and time with a surface S with surface normal $\hat{\boldsymbol{n}}$ (figure 2). The total change of mass in the volume V can be written as,

$$\frac{d}{dt}\int_V \rho dV\,, \tag{3}$$

and the total flux of mass through the surface S is,

$$-\int_S \rho \boldsymbol{u} \cdot \hat{\boldsymbol{n}} dS\,.$$

One must keep in mind that $\hat{\boldsymbol{n}}$ is the normal to the surface everywhere, pointing *outwards.* Any increase in mass takes place because of fluid mass moving normal to the surface *inwards.* Hence, the negative sign. The conservation of mass can thus be written as,

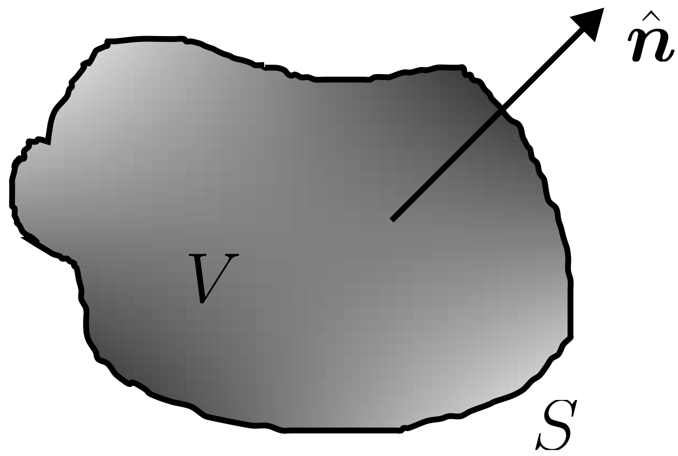

Figure 2: A control volume V with surface S and surface normal $\hat{\boldsymbol{n}}$.

$$\text{Change of mass in volume } V = \text{Net mass flowing in or out of the volume through the surface } S$$
$$\Rightarrow \frac{d}{dt}\int_V \rho dV = -\int_S \rho \boldsymbol{u}\cdot\hat{\boldsymbol{n}} dS\,. \tag{4}$$

At this point, we make use of *Gauss's divergence theorem*, which states that for any vector field $\boldsymbol{F}$ in a volume V with surface S,

$$\int_V \nabla\cdot\boldsymbol{F} dV = \int_S \boldsymbol{F}\cdot\hat{\boldsymbol{n}} dS\,. \tag{5}$$

In plain language this translates to "the sum of little changes on the inside = total big change on the outside". Using this, with $\boldsymbol{F}$ as $\rho\boldsymbol{u}$ and equation (4), we get,

$$\frac{d}{dt}\int_V \rho dV = -\int_V \nabla\cdot(\rho\boldsymbol{u}) dV\,. \tag{6}$$

Since our volume is fixed in time, we can perform the time-derivative inside the integral sign and obtain,

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right] dV = 0\,. \tag{7}$$

Since this must hold true for any arbitrary volume, the integrand itself must be zero, which gives us the equation for conservation of mass,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0\,. \tag{8}$$

1.3 Using a Lagrangian description

In a Lagrangian frame of reference, our reference frame moves with a collection of fluid particles. Thus, the volume and surface under consideration change with time. For this collection of fluid particles, mass conservation would simply be,

$$\frac{d}{dt}\int_{V(t)} \rho dV = 0\,. \tag{9}$$

Recall the Reynolds transport theorem,

$$\frac{d}{dt}\int_{V(t)} \boldsymbol{F} dV = \int_{V(t)} \frac{\partial \boldsymbol{F}}{\partial t} dV + \int_{S(t)} \boldsymbol{F}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})dS\,, \tag{10}$$

where, $\boldsymbol{F}$ can be a tensor, vector or scalar. Using it, we obtain,

$$\frac{d}{dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{S(t)} \rho(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})dS = 0\,. \tag{11}$$

As before, using the divergence theorem, we obtain,

$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right] dV = 0\,. \tag{12}$$

Since this must be true at all times for any choice of volume, the integrand must be zero, implying,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0\,. \tag{13}$$

Using vector identities, mass conservation can be written as,

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u} = 0 \\ \Rightarrow \frac{D\rho}{Dt} = -\rho(\nabla\cdot\boldsymbol{u})\,,\end{aligned} \tag{14}$$

where we have used the Lagrangian material derivative, $D\rho/Dt = \partial\rho/\partial t + \boldsymbol{u}\cdot\nabla\rho$. The above equation simply states something we could have guessed intuitively, that the change in density for a fixed collection of fluid particles occurs due to an expansion or contraction.

1.4 Special cases

Constant density

When the density is constant in space and time, $\partial\rho/\partial t = 0$ and $\nabla\rho = 0$, and we obtain the mass conservation for an *incompressible* fluid,

$$\nabla\cdot\boldsymbol{u} = 0\,. \tag{15}$$

Density constant in time

If we can ignore the time variations of density, we obtain another simplified version of mass conservation,

$$\nabla\cdot(\rho\boldsymbol{u}) = 0\,. \tag{16}$$

This is known as the ‘anelastic approximation’ and is extremely useful while studying cases where the fluid density varies in space but its rate of change (typically in the form of sound waves) takes place at a very different timescale compared to typical flow timescales. This is widely used to study interiors of gas giant planets such as Jupiter and Saturn as well as stars, including the Sun.

2 Conservation of momentum

Conservation of momentum is the statement of Newton's second law which states that the rate of change of momentum = force applied on an object. For a collection of fluid elements (Lagrangian reference frame) this can be written as,

$$\frac{d}{dt}\int_{V(t)} \rho \boldsymbol{u} dV = \int_{V(t)} \underbrace{\boldsymbol{F}_b}_{\text{Body forces}} dV + \int_{S(t)} \underbrace{\boldsymbol{\tau}}_{\text{Surface stresses}} \cdot \hat{\boldsymbol{n}} dS\,. \tag{17}$$

The most common body force on a fluid is gravity ($= \rho \boldsymbol{g}$, where $\boldsymbol{g}$ is acceleration due to gravity), but in other cases, forces such as the magnetic Lorentz force can also play a role. The surface stress tensor $\boldsymbol{\tau}$ can be further related to the strain rate tensor S using a constitutive relationship.

We first simplify the left hand side of the equation using the Reynolds transport theorem,

$$\begin{aligned} \frac{d}{dt}\int_{V(t)} \rho \boldsymbol{u} dV &= \int_{V(t)} \frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \int_{S(t)} \rho \boldsymbol{u}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})dS \\ &= \int_{V(t)} \left[\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla\cdot(\rho \boldsymbol{u}\otimes\boldsymbol{u})\right] dV\,, \end{aligned} \tag{18}$$

where we have used the divergence theorem. The last term on the RHS represents the divergence of a *dyad*, the symbol $\otimes$ represents a tensor product. We use the derivative chain rule on the first term on the RHS and the identity for the divergence of a dyadic product,

$$\nabla\cdot(\boldsymbol{a}\boldsymbol{b}) = (\nabla\cdot\boldsymbol{a})\boldsymbol{b} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}\,, \tag{19}$$

on the second term and obtain,

$$\begin{aligned} \frac{d}{dt}\int_{V(t)} \rho \boldsymbol{u} dV &= \int_{V(t)} \left[\boldsymbol{u}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right] dV \\ &= \int_{V(t)} \left[\boldsymbol{u}\underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right)}_{=0,\text{ mass conservation}} + \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)\right] dV \\ &= \int_{V(t)} \left[\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)\right] dV\,, \end{aligned} \tag{20}$$

where we have used the mass conservation equation (13). Substituting this in equation (18), we obtain,

$$\int_{V(t)} \left[\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)\right] dV = \int_{V(t)} \boldsymbol{F}_b dV + \int_{S(t)} \boldsymbol{\tau}\cdot\hat{\boldsymbol{n}} dS\,. \tag{21}$$

Using the divergence theorem yields,

$$\int_{V(t)} \left[\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)\right] dV = \int_{V(t)} \left[\boldsymbol{F}_b + \nabla\cdot\boldsymbol{\tau}\right] dV\,. \tag{22}$$

Since this holds true for all volumes at all times, we can drop the integral signs, and get,

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = \boldsymbol{F}_b + \nabla \cdot \boldsymbol{\tau} \tag{23}$$

This is known as the ***Cauchy equation*** and is the most general equation of motion. We will now derive the equation for a specific constitutive relationship between the stress and rate of strain tensors, $\boldsymbol{\tau}$ and S, respectively.

3 The Navier-Stokes equation

For a Newtonian fluid, we can relate the stress tensor $\boldsymbol{\tau}$ and rate of strain tensor S using,

$$\tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda \underbrace{S_{mm}\delta_{ij}}_{\nabla \cdot \boldsymbol{u}}, \tag{24}$$

In vector notation, $S_{mm}\delta_{ij} = \nabla \cdot \boldsymbol{u}$ and $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathbf{T}} \right)$, where

$$\nabla \boldsymbol{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}, \tag{25}$$

is the velocity gradient tensor. Considering the diagonal (isotropic) components on both sides of equation (24),

$$\tau_{ii} = -3p + (2\mu + 3\lambda)S_{mm} = -3p + (2\mu + 3\lambda)(\nabla \cdot \boldsymbol{u}), \tag{26}$$

which gives

$$p = -\frac{1}{3}\tau_{ii} + \left(\lambda + \frac{2}{3}\mu \right) (\nabla \cdot \boldsymbol{u}), \tag{27}$$

where, $\zeta = \left(\lambda + \frac{2}{3}\mu \right)$ is referred to as the *bulk* or *volume viscosity*. Here, p is the thermodynamic pressure. It is related to the *mechanical pressure* $\bar{p} = -\frac{1}{3}\tau_{ii}$ by,

$$p - \bar{p} = \left(\lambda + \frac{2}{3}\mu \right) (\nabla \cdot \boldsymbol{u}). \tag{28}$$

In the study of compressible flows, a common hypothesis that is adopted is the *Stokes assumption/hypothesis* which states that the bulk viscosity, $\zeta = 0$ and thus, making thermodynamic and mechanical pressures equivalent. It can be shown to be true for monoatomic gases and has been seen to be true for other gases and liquids. With this assumption, we can write the stress tensor as,

$$\tau_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{mm}\delta_{ij}\right) . \tag{29}$$

In vector notation,

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\left[\frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathbf{T}}\right) - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\mathbf{I}\right] , \tag{30}$$

$\mathbf{I}$ being the identity matrix.

Substituting this expression in the Cauchy equation (23), we obtain,

$$\begin{aligned}\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) &= \boldsymbol{F}_b + \nabla\cdot\left[-p\mathbf{I} + 2\mu\left[\frac{1}{2}\left((\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^{\mathbf{T}}\right) - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\mathbf{I}\right]\right] \\ \Rightarrow \rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) &= -\nabla p + \boldsymbol{F}_b + \nabla\cdot\left(2\mu\left[\frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathbf{T}}\right) - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\mathbf{I}\right]\right) .\end{aligned} \tag{31}$$

This is known as the ***Navier-Stokes equation***.

Incompressible fluid

In case of an *incompressible fluid*, $\nabla\cdot\boldsymbol{u} = 0$. Further, if the *dynamic viscosity*, μ, is assumed to be constant, we obtain a simplified version of the Navier-Stokes equation,

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \boldsymbol{F}_b + \mu\nabla^2\boldsymbol{u} . \tag{32}$$

where we have used, $\nabla\cdot(\nabla\boldsymbol{u}) = \nabla^2\boldsymbol{u}$ and $\nabla\cdot(\nabla\boldsymbol{u})^{\mathbf{T}} = \nabla(\nabla\cdot\boldsymbol{u}) = 0$. Using gravity for the body force yields a common version of the Navier stokes:

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \rho\boldsymbol{g} + \mu\nabla^2\boldsymbol{u} . \tag{33}$$

Euler equation

Ignoring viscous effects gives us the ***Euler equation***,

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \rho\boldsymbol{g} . \tag{34}$$